SUBALGEBRAS OF FREE NILPOTENT AND POLYNILPOTENT LIE ALGEBRAS

Melih Boral

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by

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Declaration

I declare that the following thesis is a record of research carried out by me, that the thesis is my own composition, and that it has not been presented previously in application for a higher degree.

Declaration

I declare that I was admitted in October 1973 under University Court Ordinance General No. 12 as a full-time Research Student in the Department of Pure Mathematics.

I certify that Melih Boral has satisfied the conditions of the Resolution and Regulations and is thus qualified to submit the accompanying thesis in application for the degree of doctor of philosophy.

Preface

I should like to express my thanks to my supervisor, Dr. Robertson, whose help and encouragement I have greately appreciated throughout the course of this work. I should also like to thank the Ministry of Education, Ankara, Turkey, for their scholarship which supported me during my studies at the University of St. Andrews.

Subalgebras of Free Nilpotent and Polynilpotent Lie Algebras

Melih Boral

Abstract : In this thesis we study subalgebras in free nilpotent and polynilpotent Lie algebras. Chapter 1 sets up the notation and includes definitions and elementary properties of free and certain reduced free Lie algebras that we use throughout this thesis. We also describe a Hall basis of a free Lie algebra as in [4] and a basis for a free polynilpotent Lie algebra which was developed in [24].

In Chapter 2 we first consider the class of nilpotency of subalgebras of free nilpotent Lie algebras starting with two-generator subalgebras. Then we study those subalgebras in a free nilpotent Lie algebra which are themselves free nilpotent. We give necessary and sufficient conditions in the case of two-generator subalgebras.

Chapter 3 extends the results obtained in Chapter 2 to the polynilpotent case. First we look at two-generator subalgebras of a free polynilpotent Lie algebra. Then we consider more general subalgebras. Finally we study those subalgebras which are themselves free polynilpotent and give necessary and sufficient conditions for two-generator subalgebras to be free polynilpotent.

In Chapter 4 we first study certain properties of ideals in free, free nilpotent and free polynilpotent Lie algebras and establish the fact that in a free polynilpotent Lie algebra a non-zero ideal which is finitely-generated as a subalgebra must be equal to the whole algebra. Then we consider the quotient Lie algebra of

a lower central term of a free Lie algebra by a term ot the lower central series of an ideal. We then generalize the results to cover the free nilpotent and free polynilpotent cases. In the last section of Chapter 4 we consider ideals of free nilpotent (and later polynilpotent) Lie algebras as free nilpotent (polynilpotent) subalgebras and establish the fact that in most non-trivial cases such an ideal cannot be free nilpotent (polynilpotent).

In the last chapter we consider the $m+k$-th term of the lower central series of a free Lie algebra as a subalgebra of the m-th term for $m \leqslant k$ and generalize the results proved in [25]. We give reasons for the failure of these results in the case $m > k$.

CONTENTS

Chapter 1

NOTATION AND TERMINOLOGY

§ 1. Basic Definitions

All Lie algebras considered in this thesis will be over arbitrary fields, denoted $\underline{k}$.

Definition 1.1 : A Lie algebra over a field $\underline{k}$ is a vector space L over $\underline{k}$ with a bilinear multiplication, denoted by (xy) for each x, y belonging to L, such that L is closed under this multiplication and satisfies :

$$\text{(i)} \quad (xx) = 0, \quad \text{for each } x \in L$$

$$\text{(ii)} \quad (x(yz)) + (z(xy)) + (y(zx)) = 0 \quad \text{for all } x, y, z \in L.$$

This last equation is called the Jacobi identity.

The symbol $\subseteq$ will denote set theoretic inclusion. If M is a subspace of L (considered as a vector space) such that M is closed under the Lie multiplication, it will be called a (Lie) subalgebra of L, denoted by $M \leqslant L$.

Let L be a Lie algebra over $\underline{k}$, and H, K be subspaces of L, considered as a vector space. Then (HK) will denote the subspace spanned by all elements of the form (hk), and $(H + K)$ will denote the subspace of all elements of the form $(h + k)$ such that $h \in H$ and $k \in K$. Obviously if H is a subspace of L, then $H \leqslant L$ if and only if $(HH) \subseteq H$.

Definition 1.2 : Let L be a Lie algebra over $\underline{k}$. If M is a subalgebra of L and $(ML) \subseteq M$, then M is called an ideal of L, denoted by $M \triangleleft L$.

By part (i) of Definition 1.1, if $x , y \in L$, then

$$(x + y) (x + y) = 0$$

and by the bilinearity of the multiplication this implies that

(*) $$(xy) = -(yx)$$

If M is an ideal of L, that is, $(ML) \subseteq M$, then by (*) we have $(LM) \subseteq M$. Hence there is no need to differentiate between right and left ideals.

Definition 1.3 : Let L be a Lie algebra over $\underline{k}$ and $M \triangleleft L$. We define the quotient algebra L / M by

$$L / M = \left\{ M + x \ : \ x \in L \right\}$$

Addition and multiplication in L / M are defined as follows :

If $x + M , y + M$ belong to L / M then

$$(x + M) + (y + M) = (x + y) + M$$

$$(x + M) (y + M) = (xy) + M$$

Only when M is an ideal in L, these operations are well-defined and obviously L / M is closed under both operations. Furthermore

$$(x + M) (x + M) = M$$

where M is the zero element in L / M . It is easy to verify that the Jacobi identity holds in L / M . Hence L / M is a Lie algebra.

Definition 1.4 : Let L , M be two Lie algebras over the same field $\underline{k}$. A homomorphism $\alpha : L \rightarrow M$ is a linear map that preserves the Lie product , i.e., if $x , y \in L$ then

$$\alpha (xy) = (\alpha(x) \alpha(y))$$

If this map is one-to-one and onto it will be called an isomorphism. If α is an isomorphism of L onto itself, it is called an automorphism of L.

The usual homomorphism theorems are true for Lie algebra homomorphisms. The proof of the following is found in [22].

Theorem 1.1 : Let L , M be Lie algebras over the same field $\underline{k}$ and $\alpha : L \to M$ be a homomorphism. Then the following is true :

(i) $\alpha(L) \leq M$; $\text{Kernel}(\alpha) \triangleleft L$, and

$$L / \text{Kernel}(\alpha) \cong \alpha(L)$$

(ii) if $H, K \triangleleft L$, $K \subseteq H$, then

$$(L/K)/(H/K) \cong L/H$$

(iii) if $H \triangleleft K$, $K \triangleleft L$, then $H \triangleleft (H + K)$,

$$H \cap K \triangleleft K \text{ and } (H+K)/H \cong K/H \cap K$$

(iv) the natural map $\phi : L \to L/H$, where $H \triangleleft L$, sets up a bijective correspondence between the ideals (subalgebras) of L/H and ideals (subalgebras) of L between H and L.

§ 2. Free Lie Algebras

In this thesis we will mainly be concerned with free or reduced free Lie algebras.

Definition 1.5 : Let X be an arbitrary set. A free Lie algebra on X is a pair (F, i) , where F is a Lie algebra and $i : X \to F$ is a map such that if $\alpha : X \to B$ is any map of X into a Lie algebra B , then there exists a unique Lie homomorphism $\eta : F \to B$ such that $\alpha = \eta i$.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "\alpha"'] & F \arrow[dl, dashed, "\eta"] \\ B & \end{tikzcd}$$

We now construct a free Lie algebra on a set X. For each positive integer n we define a set X_n as follows :

$$X_1 = X$$

$$(*) \qquad X_n = \bigcup_{p=1}^{n-1} (X_p \times X_{n-p})$$

where " $\times$ " denotes the Cartesian product of sets. Let $M(X) = \bigcup_{n=1}^{\infty} X_n$. For any a , $b \in M(X)$, there exists integers p , q such that $a \in X_p$ and $b \in X_q$ so that $(a,b) \in X_p \times X_q$. Let $n = p + q$. Then $(a,b) \in X_p \times X_{n-p}$ which is one of the constituents of the union $(*)$ for X_n. Denote by (ab) the image of (a,b) under the canonical injection of $(X_p \times X_{n-p})$ into X_n. Hence, for any a , $b \in M(X)$, we can define a product (ab) as described above. Call the integer p, such that $a \in X_p$, the length of the element a of $M(X)$, denoted by $\ell(a)$. We have

$$\ell(ab) = \ell(a) + \ell(b)$$

Elements of length one are the elements of X. For elements of length ≥ 2, we necessarily have

$$c = (ab)$$

where a and b have smaller lengths than c.

Let $\underline{k}$ be an arbitrary field. Form a vector space over $\underline{k}$ with basis $M(X)$, that is, take all $\underline{k}$-linear combinations of elements in $M(X)$ and extend the multiplication of $M(X)$ to the whole of this vector space. The result will be a free non-associative algebra $N(X)$ over $\underline{k}$, which is not finite-dimensional as a vector space. In other words, $N(X)$ is a $\underline{k}$-module with basis $M(X)$ and a non-associative multiplication.

Now, let A be the ideal in $N(X)$ generated by all elements of the form

$$Q(a) = (aa)$$

$$J(a,b,c) = a(bc) + c(ab) + b(ca)$$

Of course, A is an ideal in a non-associative algebraic sense. Then

$$F(X) = N(X) / A$$

is the free Lie algebra on X.

The set X is called a set of free generators for $F(X)$. When there is no confusion about which set $F(X)$ is constructed on, we will simply write F for $F(X)$.

Theorem 1.2 : Let $\eta : N(X) \rightarrow F(X)$ be the canonical mapping and ϕ be the restriction of η to X. For every mapping $f : X \rightarrow L$ where L is an arbitrary Lie algebra, there exists a Lie homomorphism $g : F(X) \rightarrow L$ such that $f = g\phi$ and the mapping g is unique.

A detailed discussion of free Lie algebras and the proof of this theorem is given in [4].

Definition 1.6 : Let L be an arbitrary Lie algebra and $A = \{ a_i : i \in I \}$ be a family of elements of L. Let $F(I)$ be the free Lie algebra constructed on the set I and σ be the homomorphism from $F(I)$ into L such that

$$\sigma : i \longmapsto a_i$$

If σ is surjective, then A is called a generating set for L. If σ is bijective, then L is a free Lie algebra and A is a set of free generators for L. If A is a finite set , L is said to be finitely generated.

According to this definition any two free generating sets in a free Lie algebra F have the same cardinality. We will call the cardinality of a free generating set of F the rank of F.

Let F be a free Lie algebra on free generators X over a field $\underline{k}$. Any automorphism α maps X to another free generating set $\alpha(X)$ of F and α is completely determined by its effect on X. Conversely, any bijection from one free generating set of F to another determines a unique automorphism of F.

Definition 1.7 : Let Y be any finite subset of F. Then the following are called elementary Lie transformations :

(i) applying a non-singular linear transformation to the elements of Y

(ii) replacing any $y \in Y$ by $y + f(y_1,\dots,y_m)$ where f is an expression of F in the elements of Y distinct from y.

Any elementary Lie transformation applied to a free generating set of F defines an automorphism.

In the case F is finitely generated, any automorphism can be obtained this way as the following theorem (proved in [5]) indicates :

Theorem 1.3 : Let F be a free Lie algebra on a finite generating set X. Then every automorphism of F may be obtained by applying to X a succession of elementary Lie transformations.

In group theory Schreier's theorem states that every subgroup of a free group is free. A similar theorem for Lie algebras is proved by A. I. Širšov [17] which states that every subalgebra of a free Lie algebra is itself a free Lie algebra. We will use this result without further reference.

§ 3. Hall Basis of a Free Lie Algebra

The set $M(X)$ defined in the previous section is a basis for the non-associative free algebra $N(X)$ considered as a vector space over $\underline{k}$. In $F(X)$, however, $M(X)$ will not be linearly independent, since elements like (ab) and (ba), where $a , b \in M(X)$, which are linearly independent in $N(X)$, are linearly dependent in $F(X)$ due to

$$(ba) = -(ab)$$

Similar relations arise in $F(X)$ because of the Jacobi identity. Hence $M(X)$ is not a basis for $F(X)$.

In this section we construct a basis for the free Lie algebra $F(X)$, considered as a vector space.

Let $M^n(X)$ denote those elements in $M(X)$ of length n. Obviously $M^1(X) = X$.

Definition 1.8 : A Hall set $H \subseteq M(X)$ is defined as follows :

(i) $X \subseteq H$, and X is given an arbitrary total ordering,

(ii) $H \cap M^2(X)$ consists of elements (xy) such that $x , y \in X$ and $x < y$,

(iii) suppose $H \cap M^m(X)$ are defined and given a length-preserving ordering, for $m = 1,2,\ldots,n-1$, i.e., if $u , v \in M(X)$ and $\text{length}(u) < \text{length}(v)$, put $u < v$, and order the elements of same length arbitrarily. Then, for $n \geq 3$, we define $H \cap M^n(X)$ to consist of all elements in $M^n(X)$ of the form $a(bc)$ where a , b , c , bc belong to $\bigcup_{k=1}^{n-1} (H \cap M^k(X))$ and satisfy

$$a \geq b < c$$

$$a < bc$$

Now put $H = \bigcup_{n=1}^{\infty} (H \cap M^n(X))$. (We write the product as bc for convenience)

Let us use the following notation :

$$H_n = H \cap M^n(X)$$

Then, $H = \bigcup_{n=1}^{\infty} H_n$. For a given set X one can construct different Hall sets, each one determined by the ordering given to it. We now write in a more precise form

$H_1 = X$, X is given a total order.

$H_2 = \left\{ (xy) \ : \ x , y \in X \ ; \ x < y \right\}$

$$H_n = \left\{ a(bc) \ : \ a , b , c , bc \in (H_1 \cup H_2 \cup \ldots \cup H_{n-1}) \ ; \ a \geqslant b < c, \ a < bc \right\}$$

Theorem 1.4 : Let F be the free Lie algebra on a set X. Then, a Hall set constructed on X forms an additive basis for the Lie algebra F considered as a vector space.

The proof of this theorem is given in [4].

Once a Hall set is constructed with a definite ordering, we keep the same ordering. Henceafter, whenever an element of the free Lie algebra F is mentioned, its representation as a linear combination of these basis elements is intended, unless stated otherwise.

Note that a Hall set H on X is a linearly independent set in $F(X)$, and H_n consists of those elements in H of length n. In this thesis we fix the following ordering of the set H :

(i) X is ordered arbitrarily (usually consistent with the indices used to denote its letters)

(ii) Let $u = u_1u_2$, $v = v_1v_2$ be elements of H. If length(u) < length(v), put $u < v$. If u and v have the same length, then put $u < v$ if and only if either $u_2 < v_2$ or $u_2 = v_2$ and $u_1 < v_1$.

Example 1.1 : Let $X = \{ a , b , c \}$. Then

H_1 = X and suppose $a < b < c$.

$H_2 = \{ ab , ac , bc \}$ and suppose $ab < ac < bc$.

$H_3 = \{ a(ab) , b(ab) , c(ab) , a(ac) , b(ac) , c(ac) , b(bc) , c(bc) \}$ and suppose they are ordered as they are written.

$H_4 = \{ a(a(ab)) , b(a(ab)) , c(a(ab)) , b(b(ab)) ,$

$c(b(ab))$, $c(c(ab))$, $a(a(ac))$, $b(a(ac))$, $c(a(ac))$
$b(b(ac))$, $c(b(ac))$, $c(c(ac))$, $b(b(bc))$, $c(b(bc))$
$c(c(bc))$, $(ab)(ac)$, $(ab)(bc)$, $(ac)(bc)$ $\}$

If X is a finite set, there is a method of determining the number of elements in H_n. First we need the following :

Definition 1.9 : Let N denote the set of positive integers. The Möbius function $\mu : N \to \{1, 0, -1\}$ is defined as follows :

$\mu(n) = 0$ if n is divisible by the square of a prime number, i.e., if n has a square factor.

$\mu(1) = 1$

$\mu(n) = (-1)^k$ if n is not divisible by square of a prime number and $n = p_1, \dots, p_k$ are prime divisors of n.

Example 1.2 : $\mu(1) = 1$, $\mu(2) = -1$, $\mu(3) = -1$ $\mu(4) = 0$, $\mu(5) = -1$, $\mu(6) = 1$, ...

Theorem 1.5 : Let X be a set and $|X| = r$ denote the number of elements in X. Then, if H is a Hall set constructed on X, the number of elements in H_n, $n \geq 1$, is given by

$$1/n \sum_{\substack{d \\ d \text{ divides } n}} \mu(d)\, r^{n/d}$$

where μ is the Möbius function.

The proof of this theorem is given in [4].

Example 1.3 : Suppose $|X| = 3$. Then

$|H_1| = 3$

$|H_2| = 3$

$$|H_3| = 8$$
$$|H_4| = 18$$
$$|H_5| = 48$$

and in general the number of elements in H_n is given by

$$1/n \sum_{\substack{d \\ d \text{ divides } n}} \mu(d)\, 3^{n/d}$$

Corresponding to a Hall basis for free Lie algebras, there is the theory of basic commutators in free groups, which is studied in detail in [23]. Lie algebra basis and related topics are studied in [4] and [19].

Note that the free Lie algebra on one generator is an exceptional case, since from $(xx) = 0$ it follows that as a vector space it is one-dimensional. Most of the results of this thesis become trivial when considered in the context of a free Lie algebra on one generator. Hence throughout this thesis when we mention a free Lie algebra on free generators X, we assume that $|X| \geq 2$.

Let F be a free Lie algebra on the free generating set X and with a Hall basis H. Then there are proper subalgebras of F which are isomorphic to F.

<u>Example 1.4</u> : Let F be as above and suppose $|X| \geq 2$. If $|X| = n$, then $|H_i| \geq n$ for all $i = 1,2,\ldots$. Take any n elements in H_2, say $\{h_1,\ldots h_n\}$. If $X = \{x_1,\ldots,x_n\}$ then the map

$$\vartheta : x_i \to h_i$$

can be extended to a monomorphism.

In fact, given any positive integer $m \geq 1$, there is a subalgebra of F of rank m, since for any such m there exists an integer v

such that $|H_v| \geqslant m$. As we will see later the elements of H_v freely generate a free Lie subalgebra of F.

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§ 4. Lower Central Series and Nilpotent Lie Algebras

<u>Definition 1.10</u> : Let L be a Lie algebra over a field $\underline{k}$. We define the lower central series $\{ L_{(n)} \}$ of L as follows :

$$L_{(1)} = L$$

$$L_{(n+1)} = (LL_{(n)})$$

If there is an integer q such that $L_{(q-1)} \neq \{0\}$, but $L_{(q)} = \{0\}$, L is called nilpotent of class q.

A Lie algebra L is called abelian if it is nilpotent of class 2, that is, if $(LL) = 0$.

It is easy to verify that for each n, the n-th term of the lower central series of L, $L_{(n)}$, is an ideal in L. in fact,

$$L_{(n)} \triangleleft L_{(n-1)} \triangleleft \cdots \triangleleft L_{(2)} \triangleleft L_{(1)} = L$$

Hence one can construct lower central factors $L_{(n)}/L_{(n+1)}$ and $L / L_{(n)}$.

As an easy consequence of the Jacobi identity we have

$$(*) \qquad (L_{(m)}L_{(n)}) \subseteq L_{(m+n)}$$

This fact will be used without further reference.

If L is a nilpotent Lie algebra, then any subalgebra of L will also be nilpotent possibly of a lower class than the original algebra.

We have stated in the previous section that given a set X, the Hall set H on X forms an additive basis for the free Lie algebra F on X. By (*) above, any element in F of length $\geq n$ is contained in $F_{(n)}$. Hence, $H_n \subseteq F_{(n)}$.

A detailed version of the theorems stated below is found in [23] for Lie algebras. Analogues for groups and Lie rings is found in [10] and [18].

Theorem 1.6 : Let F be a free Lie algebra over a field $\underline{k}$ and let H be a Hall basis for F. If f is any element in F, then

$$f = \sum_{i=1}^{r} \alpha_i h_i \quad (\text{modulo } F_{(n)})$$

where $\alpha_1,\ldots,\alpha_r$ are elements of the field $\underline{k}$, and $h_1,\ldots,h_r$ are elements of $(H_1 \cup \ldots \cup H_{n-1})$.

Definition 1.11 : A Lie algebra L is called free nilpotent on free generators X, if there is a free Lie algebra F on the same generating set X such that

$$L = F / F_{(n)}$$

for some positive integer n which is called the class of nilpotency of L.

Theorem 1.7 : Let F be a free Lie algebra over a field $\underline{k}$ and H be a Hall basis for F. Then, $F_{(n)}/ F_{(n+1)}$ is a free abelian Lie algebra and the elements of length n in H form a basis for it.

Theorem 1.8 : Let $L = F / F_{(n)}$ be a free nilpotent and H be a Hall basis for F. Then, the elements in H of length $< n$, that is, $(H_1 \cup \ldots \cup H_{n-1})$ form an additive basis for L.

Note that in a free nilpotent Lie algebra the only additional relation is that of being nilpotent, that is, if F is a free Lie algebra on X, and R is the ideal generated by all elements of length $\geq n$, for some positive integer n, then F / R is a free nilpotent Lie algebra.

§ 5. Free Generators for the Terms of the Lower Central Series of a Free Lie Algebra

The terms of the lower central series $F_{(m)}$ of a free Lie algebra over $\underline{k}$ are not finitely generated as subalgebras of F, for $m \geq 2$. The only exception is the one-generator free Lie algebra which is free abelian. The problem of finding free generating sets for terms of the lower central series was first posed in the context of group theory by Grünberg [8]. A corresponding result was stated for free Lie rings in [27]. Free generating sets for $F_{(m)}$, where F is a free Lie algebra are given by A. L. Šmel'kin in [20]. A very detailed version for groups and free Lie rings is also found in [24]. In this section we will briefly describe their methods.

Let F be a free Lie algebra over a field $\underline{k}$ with a free generating set X. Let H be a Hall basis for F constructed on X and $H_{(n)}$ denote the set of elements in H of length n. H is ordered in a length preserving manner as in § 3. Let $F_{(m)}$ denote the m-th term of the lower central series of F. The following result is proved in [20] :

Theorem 1.9 : The set C_m defined as

$$C_m = \left\{ x = a_1 a_2 \ : \ a_1, a_2 \in H \ ; \ \text{length}(x) \geq m \ ; \ x \in H \ ; \ \text{length}(a_1) < m \right\}$$

is a set of free generators for $F_{(m)}$.

Using the definition of H, we can express $C_{(m)}$ more explicitly as follows :

$$C_m = \left\{ \begin{array}{l} x = x_1(x_2(\dots (x_{r-1}x_r))\dots) \ : \ \text{length}(x) \geq m \ ; \\ x_i \in (H_1 \cup \ \dots \ \cup H_{m-1}) \ ; \ x_1 \geq x_2 \geq \dots \geq x_{r-1} < x_r \ ; \\ r \geq 2 \ ; \ \text{ if } \ x_r = b_1 b_2, \text{ then } \ x_{r-1} \geq b_1 \end{array} \right\}$$

We now construct a Hall basis H^{C_m} on free generators C_m for the Lie algebra $F_{(m)}$, by forming products of elements of C_m. If h is a product of elements form C_m, we will refer to C_m-length and X-length of h meaning the number of letters used from C_m or X respectively.

C_m is a subset of H, so it can be given an order which coincides with the order in H. Let

$$H_1^{C_m} = C_m$$

$$H_2^{C_m} = \left\{ a_1 a_2 \ : \ a_1, a_2 \in C_m \ ; \ a_1 < a_2 \right\}$$

We now give $H_2^{C_m}$ an order as follows : Let h , $g \in H_2^{C_m}$, where $h = h_1 h_2$ and $g = g_1 g_2$, $h_1, h_2, g_1, g_2 \in C_m$. If X-length($h$) is less than X-length($g$), put $h < g$. Suppose h and g have the same X-length. Then put $h < g$ if and only if either $h_2 < g_2$ or $h_2 = g_2$ and $h_1 < g_1$.

Suppose $H_1^{C_m}, \dots, H_{n-1}^{C_m}$ are defined and ordered. We put

$$H_n^{C_m} = \left\{ \begin{array}{l} x = a_1(a_2 a_3) \ : \ C_m\text{-length}(x) = n \ ; \ a_1 < a_2 a_3 \ ; \\ a_1 \geq a_2 < a_3 \ ; \ a_1, a_2, a_3, a_2 a_3 \in (H_1^{C_m} \cup \ \dots \ \cup H_{n-1}^{C_m}) \end{array} \right\} ,$$

where the inequality signs refer to the ordering in $(H_1^{C_m} \cup \dots \cup H_{n-1}^{C_m})$. We now give $H_n^{C_m}$ an ordering similar to the one described for $H_2^{C_m}$. This ordering can now be extended to H^{C_m} which is defined as

$$H^{C_m} = \bigcup_{j=1}^{\infty} H_j^{C_m}$$

Note that the ordering of H^{C_m} need not (and is not) be compatible with that of H. Obviously, there are elements in $H_2^{C_m}$ which have X-length less than an element of $H_1^{C_m} = C_m$.

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§ 6. Polycentral Series and Polynilpotent Lie Algebras

<u>Definition 1.12</u> : Let L be a free Lie algebra over a field $\underline{k}$, and let $\{L_{(n)}\}$, $n \in N$, be the lower central series of L. Suppose $\{n_1,\dots,n_i,\dots\}$ is an arbitrary sequence of integers, $n_i \geq 1$. We define the polycentral series

$$L \supseteq L_{(n_1)} \supseteq \dots \supseteq L_{(n_1),(n_2),\dots,(n_i)} \supseteq L_{(n_1),\dots,(n_{i+1})}$$

as follows : $L_{(n_1)}$ is the n_1-th term of the lower central series of L, $L_{(n_1),\dots(n_{i+1})} = (L_{(n_1),\dots,(n_i)})_{(n_{i+1})}$ is the (n_{i+1})-th term of the lower central series of $L_{(n_1),\dots,(n_i)}$.

L is called polynilpotent Lie algebra relative to the sequence $\{n_1,\dots,n_k\}$ if

$$(*) \qquad L_{(n_1),\dots,(n_k)} = \{0\}$$

where none of the integers n_i, $i = 1,2,\dots,k$, can be replaced by a smaller positive integer and an equation of the form $(*)$ still be true.

The terms of the polycentral series of L form a chain of ideals in L

$$L_{(n_1),\dots,(n_{i+1})} \triangleleft L_{(n_1),\dots,(n_i)} \triangleleft \dots \triangleleft L_{(n_1)} \triangleleft L$$

Hence one can construct the polycentral factor algebras

$$L_{(n_1),\dots,(n_i)} / L_{(n_1),\dots,(n_{i+1})}$$

If L is a polynilpotent Lie algebra relative to some sequence of integers, then any subalgebra of L will also be polynilpotent, possibly relative to a different sequence of integers.

Definition 1.13 : A Lie algebra L is called free polynilpotent relative to the sequence $\{n_1,\dots,n_k\}$ on free generators X if there exists a free Lie algebra F on free generators X such that

$$L = F / F_{(n_1),\dots,(n_k)}$$

for a sequence of positive integers $\{n_1,\dots,n_k\}$.

Note that in the definitions given above, if all the integers $n_1 = n_2 = \dots = n_k = 1$, we have

$$L_{(n_1),\dots,(n_k)} = L$$

If $k = 1$, then

$$L = F / F_{(n_1)}$$

is a free nilpotent Lie algebra of class n_1.

A group theoretic analogoue of the following theorem is proved in [20].

Theorem 1.9 : In a free polynilpotent Lie algebra L, the factors $L_{(m)} / L_{(m+1)}$ of the lower central series are free abelian Lie algebras.

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§ 7. Free Generators and Bases For the Terms of the Polycentral Series of a Free Lie Algebra

Let F be a free Lie algebra over a field $\underline{k}$ with a free generating set X. Let H be a Hall basis for F constructed on X. In this section, we describe a free generating set for $F_{(n_1),\dots,(n_k)}$ for some sequence of integers $\{ n_1,\dots,n_k \}$, and then construct a Hall basis for this subalgebra of F.

In § 5 we defined a free generating set C_{n_1} for $F_{(n_1)}$ as follows :

$$C_{n_1} = \left\{ x = a_1 a_2 \ : \ x \in H \ ; \ \text{length}(x) \geq n_1 \ ; \ \text{length}(a_1) < n_1 \right\}$$

We then constructed a Hall basis $H^{C_{n_1}}$ for $F_{(n_1)}$.

Consider $F_{(n_1),(n_2)}$ as a free subalgebra of $F_{(n_1)}$. Proceeding in the same manner we define a free generating set C_{n_1,n_2} as follows :

$$C_{n_1,n_2} = \left\{ x = a_1 a_2 \ : \ x \in H^{C_{n_1}} \ ; \ C_{n_1}\text{-length}(x) \geq n_2 \ ; \ C_{n_1}\text{-length}(a_1) < n_2 \right\}$$

We order C_{n_1,n_2} as follows : Let g , $h \in C_{n_1,n_2}$. If C_{n_1}-length$(g) <$ C_{n_1}-length(h), put $g < h$. If C_{n_1}-length$(g) = C_{n_1}$-length(h) and X-length$(g) < X$-length(h), then again we put $g < h$. Suppose both C_{n_1}-length$(g) = C_{n_1}$-length(h) and X-length$(g) = X$-length(h). Then put $g < h$ if either $g_2 < h_2$ or $g_2 = h_2$ and $g_1 < h_1$, where $g = g_1 g_2$ and $h = h_1 h_2$.

We now construct a Hall basis $H^{C_{n_1,n_2}}$ for $F_{(n_1),(n_2)}$ in the usual manner,

$$H_1^{C_{n_1,n_2}} = C_{n_1,n_2}$$

$$H_2^{C_{n_1,n_2}} = \left\{ a_1 a_2 \ : \ a_1, a_2 \in C_{n_1,n_2} \ ; \ a_1 < a_2 \right\}$$

and we give this set an ordering as in § 5. Suppose $H_1^{C_{n_1,n_2}},\dots,H_{m-1}^{C_{n_1,n_2}}$ are defined and ordered. Then,

$$H_m^{C_{n_1,n_2}} = \Big\{ x = a_1(a_2a_3) \ : \ C_{n_1,n_2}\text{-length } (x) = m \ ; \ a_1 < a_2a_3$$
$$a_1 \geqslant a_2 < a_3 \ ; \ a_1, a_2, a_3, a_2a_3 \in (H_1^{C_{n_1,n_2}} \cup \ldots \cup H_{m-1}^{C_{n_1,n_2}})$$

Now suppose that free generating sets and Hall bases for $F_{(n_1)}, \ldots,$ $F_{(n_1),\ldots,(n_{k-1})}$ are defined and ordered. We define a free generating set for $F_{(n_1),\ldots,(n_k)}$ as follows :

$$C_{n_1,\ldots,n_k} = \Big\{ x = a_1a_2 \ : \ x \in H^{C_{n_1,\ldots,n_k}} \ ; \ C_{n_1,\ldots,n_{k-1}}\text{-length } (x) \geqslant n_k \ ; \ C_{n_1,\ldots,n_{k-1}}\text{-length } (a_1) < n_k \Big\}$$

It is easy to see that this set is a free generating set for $F_{(n_1),\ldots,(n_k)}$, since if $G = F_{(n_1),\ldots,(n_{k-1})}$, then $F_{(n_1),\ldots,(n_k)} = G_{(n_k)}$ and the result follows from Theorem 1.9 .

One constructs a Hall set on $C_{n_1,\ldots,n_k}$ in the usual manner. Put

$$H_1^{C_{n_1,\ldots,n_k}} = C_{n_1,\ldots,n_k}$$

This set is ordered by considering the following in decreasing order of priority

$C_{n_1,\ldots,n_{k-1}}$-lengths
$C_{n_1,\ldots,n_{k-2}}$-lengths
$\vdots$
C_{n_1}-lengths
X-lengths

If two elements of $H^{C_{n_1,\ldots,n_k}}$ have the same length with respect to every one of the generating sets above, we then use the method of ordering described previously (i.e., in § 3).

Suppose $H_1^{C_{n_1,\ldots,n_k}}, \ldots, H_{m-1}^{C_{n_1,\ldots,n_k}}$ are defined and ordered. Then

$$H_m^{C_{n_1,\dots,n_k}} = \left\{ x = a_1(a_2a_3) \ : \ C_{n_1,\dots,n_k}\text{-length}(x) \geq m \ ; \ a_1,a_2,a_3,a_2a_3 \in \left(H_1^{C_{n_1,\dots,n_k}} \cup \dots \cup H_{m-1}^{C_{n_1,\dots,n_k}} \right) \ a_1 \geq a_2 < a_3 \ ; \ a_1 < a_2a_3 \right\}$$

and we order the elements of $H^{C_{n_1,\dots,n_k}}$ by giving priority to $C_{n_1,\dots,n_k}$-lengths.

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§ 8. Bases for Free Polynilpotent Lie Algebras

A basis for a free polynilpotent group was first given by M. Ward in [24], where all statements were proved in detail. A shorter and less detailed version appeared in a paper by L. A. Bokut', [3], for Lie algebras. In this section we describe a basis for a free polynilpotent Lie algebra using the notation of the previous section.

<u>Definition 1.14</u> : Let $L = F / F_{(n_1),\dots,(n_k)}$ be a free polynilpotent Lie algebra relative to the sequence $\{n_1,\dots,n_k\}$, where F is a free Lie algebra on a free generating set X.
Let $C_{n_1,\dots,n_i}$ and $H^{C_{n_1,\dots,n_i}}$ be the free generating set and the Hall set on $F_{(n_1),\dots,(n_i)}$, $i < k$, as described in the previous section. Suppose we denote the set of elements in $H^{C_{n_1,\dots,n_i}}$ of $C_{n_1,\dots,n_i}$-length m by $H_m^{C_{n_1,\dots,n_i}}$. For $i = 1,2,\dots,k$ we define the sets B_i as follows :

$$B_1 = H_1 \cup H_2 \cup \dots \cup H_{n_1-1}$$

$$B_2 = H_1^{C_{n_1}} \cup H_2^{C_{n_1}} \cup \dots \cup H_{n_2-1}^{C_{n_1}}$$

$$\vdots$$

$$B_k = H_1^{C_{n_1,\dots,n_{k-1}}} \cup \dots \cup H_{n_k-1}^{C_{n_1,\dots,n_{k-1}}}$$

We put $B = \bigcup_{i=1}^{k} B_i$.

The sets B_i are mutually disjoint, that is, $B_i \cap B_j$ is empty when $i \neq j$, $i , j = 1,2,\ldots,k$. Note that if $b \in B$, then $b \neq 0$ in L, since $C_{n_1,\ldots,n_{k-1}}$-length $(b) < n_k$. Suppose $f \in F_{(n_1),\ldots(n_i)}$ and $f \notin F_{(n_1),\ldots,(n_{i+1})}$ then

$$f = \sum_j \alpha_j h_j \quad (\text{modulo } (F_{(n_1),\ldots,(n_{i+1})}))$$

where $\alpha_j \in \underline{k}$ and $h_j \in H^{C_{n_1,\ldots,n_i}}$. Suppose

$$C_{n_1,\ldots,n_i}\text{-length } (h_j) \geqslant n_{i+1}$$

Then, $h_j \in F_{(n_1),\ldots,(n_{i+1})}$. Hence the h_j used in the expression for f above belong to $(H_1^{C_{n_1,\ldots,n_i}} \cup \ldots \quad H_{n_{i+1}-1}^{C_{n_1,\ldots,n_i}}) = B_{i+1}$. Thus the set B_{i+1} forms an additive basis for $F_{(n_1),\ldots,(n_i)} / F_{(n_1),\ldots,(n_{i+1})}$.

We now state :

<u>Theorem 1.11</u> : Let $L = F / F_{(n_1),\ldots,(n_k)}$ be a free poly-nilpotent Lie algebra relative to the sequence $\{ n_1,\ldots,n_k \}$. Then, the set B defined above forms an additive basis for L, that is, if $f \in L$, then

$$f = \sum \beta_j h_j$$

where $\beta_j \in \underline{k}$ and $h_j \in B$.

A proof of this theorem is found in [24].

The basis B of L is ordered by considering

$C_{n_1\ldots n_{k-1}}$-lengths

$C_{n_1\ldots n_{k-2}}$-lengths

$\vdots$

C_{n_1}-lengths

X-lengths

in decreasing order of priority.

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§ 9. Soluble Lie Algebras

Definition 1.15 : Let L be a Lie algebra over a field $\underline{k}$. We define the derived series of L, $\{\delta^n L\}$,

$$L = \delta^0 L \supseteq \delta^1 L \supseteq \ldots \supseteq \delta^m L \supseteq \delta^{m+1} L \supseteq \ldots ,$$

where $\delta^1 L = L_{(2)}$ and $\delta^m L = L_{\underbrace{(2),(2),\ldots,(2)}_{m\text{-times}}}$.

L is called a soluble Lie algebra if $\delta^n L = \{0\}$ for some integer n. If n is the smallest integer such that for every integer $k \geqslant n$ we have $\delta^k L = \{0\}$, then n is called the derived length of L. If $n = 2$, we say that L is metabelian.

Note that soluble Lie algebras can be considered as a special case of the polynilpotent Lie algebras, by putting $n_1 = n_2 = \ldots = n_k = 2$ in the definitions of § 6. All the results mentioned in § 7 and § 8 may be considered for the free soluble case. (We say L is a free soluble Lie algebra if there is a free Lie algebra F on the same generating set as L such that $L = F / \delta^k L$ for some positive integer k). In particular if $L = F / \delta^k L$ is a free soluble Lie algebra, the set $B = B_1 \cup B_2 \cup \ldots \cup B_k$ where

$$\begin{aligned} B_1 &= X \\ B_2 &= C_2 \\ &\vdots \\ B_k &= C_{\underbrace{2,2,\ldots,2}_{(k-1)\text{-times}}} \end{aligned}$$

forms an additive basis for L. Obviously $C_{\underbrace{2,2,\ldots,2}_{i\text{-times}}}$ is a basis for the factor algebra $\delta^{i}F / \delta^{i+1}F$.

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§ 10. Connection Between Lie Algebras (Lie Rings) and Groups

Many of the results proved in this thesis for Lie algebras (and Lie rings) are also true for groups. One can define mappings between those algebraic structures. A method of associating a Lie ring to a group was considered by E. Witt in [26] and [27] , and it was studied in detail by M. Lazard [12].

Let us first recall the definition of a Lie ring :

<u>Definition 1.16</u> : A Lie ring L is a non-empty set with two binary operations , addition , + , and multiplication , ( ), such that relative to addition L is an abelian group and relative to multiplication we have left and right distributivity, and for all $a , b , c \in L$

$$(aa) = 0$$

$$(a (bc)) + (c (ab)) + (b (ca)) = 0$$

A Lie ring can be considered as a Lie algebra where "scalars" are integers.

Let G be a group with a free generating set $X = \{x_1,\dots,x_i,\dots\}$ and for $a , b \in G$ define a commutator

$$[a,b] = a^{-1}b^{-1}ab$$

where we use the multiplicative notation for the group operation. Let $G_{(n)}$ be the n-th term of the lower central series of G, that is , the subgroup of G generated by all commutators of length n and higher. (For various terms used here one can refer to [9]).

$G_{(n)}$ is a normal subgroup of G and one can form the factor groups $G_{(n)} / G_{(n+1)}$, for $n \geq 1$.

Let L be the free Lie ring on free generators X. There is a correspondence between elements of a free group G and L if we put

in G	in L
ab	a + b
[a,b]	ab

Then, $G_{(n)} / G_{(n+1)}$ is isomorphic to the additive group of homogeneous elements of length n in L. This last isomorphism is called Witt isomorphism.

The connections between nilpotent Lie algebras over the field of rational numbers and nilpotent groups is studied by K. K. Andreev in [1].

In general one cannot form an isomorphism between $G_{(n)} / G_{(n+1)}$, where G is a free group, and the additive group of homogeneous elements of a free Lie algebra F, but there is a one-to-one correspondence between basic commutators of a free group as described in [10] and a Hall basis of a free Lie algebra F whose generating set has the same cardinality as that of G. However, the correspondence between free generating sets for $G_{(n)}$ and those of $F_{(n)}$, for $n \geq 2$, is not one-to-one. In the group theory case, for free generating sets of $G_{(n)}$, one has to define a particular type of expression, which is described in [24] as a "commutator sprinkled with inverses". The theory developed in [23] and [24] applies to Lie algebras in a more simple form.

Chapter 2

ON SUBALGEBRAS OF FREE NILPOTENT LIE ALGEBRAS

A subalgebra of a nilpotent Lie algebra is itself nilpotent possibly of a lower class than the original algebra. In the first section of this chapter we determine the class of nilpotency of two generator subalgebras of a free nilpotent Lie algebra. The next section is devoted to the study of subalgebras which are not necessarily on two generators. In section three we look at those subalgebras of a free nilpotent Lie algebra which are themselves free nilpotent.

§ 1. Two-Generator Subalgebras of a Free Nilpotent Lie Algebra

Let F be a free Lie algebra over a field $\underline{k}$ with a free generating system X, where $|X| \geq 2$. Let $L = F / F_{(p)}$ be a free nilpotent Lie algebra of class p, for some positive integer $p \geq 2$.

<u>Definition 2.1</u> : Let $f \in L = F / F_{(p)}$ such that

$$f = \alpha_1 h_1 + \dots + \alpha_r h_r$$

where $0 = \alpha_i \in \underline{k}$, $h_i \in (X \cup H_2 \cup \dots \cup H_{p-1})$ which is an additive basis for L. Suppose h_1 is the minimal element in the set $\{h_1, \dots, h_r\}$, where minimality is with respect to the order in H. We call h_1 the leading term of f and denote it by $\mathrm{ld}\,(f)$.

<u>Theorem 2.1</u> : Let S be a two generator subalgebra of L on a generating set $Y = \{y_1, y_2\}$ such that $S \subseteq F_{(v)}$ but $S \not\subseteq F_{(v+1)}$, where $v < p$. Suppose Y is linearly independent modulo $F_{(q+1)}$ and q is the smallest such integer, where

$$1 \leq v \leq q < p$$

Then S is nilpotent of class s, where s is the smallest integer such that

$$(s-1)v + q \geqslant p$$

<u>Proof</u> : Let $Y = \{y_1, y_2\}$ be a generating set for S and suppose $\mathrm{ld}(y_1) = h_1 \in H_v$, which must be the case since $S \subseteq F_{(v)}$ but $S \not\subseteq F_{(v+1)}$. Let q be as described above. If $\mathrm{ld}(y_2) = h_2$ and $h_2 \in H_w$, $h_2 \neq h_1$, $v \leqslant w$, then $q = w$. If $h_2 = h_1$, then there exists $h \in H_q$ occuring in y_2 such that the leading term of y_1y_2 is h_1h, which is of X-length $v + q$. (see note at the end of this proof).

In $S_{(s)}$, where s is as in the statement of the theorem, an element whose leading term has minimal X-length is of the form

$$f = \alpha \underbrace{y_1(y_1(\ldots(y_1}_{(s-1)\text{-times}}y_2))\ldots) + \ldots$$

where $\alpha \in \underline{k}$ and

$$\mathrm{ld}(f) = h_1(h_1(\ldots(h_1h))\ldots).$$

Then

$$X\text{-length}(\mathrm{ld}(f)) = (s-1)v + q \geqslant p$$

Hence, $f = 0$ in L. Since for any $g \in S_{(s)}$

$$X\text{-length}(\mathrm{ld}(g)) \geqslant X\text{-length}(\mathrm{ld}(f)) \geqslant p$$

we conclude that $S_{(s)} = \{0\}$ in L. Similarly for any $t \geqslant p$, we have $S_{(t)} = \{0\}$.

In $S_{(s-1)}$, however, if

$$f = \alpha \underbrace{y_1(y_1(\ldots(y_1}_{(s-2)\text{-times}}y_2))\ldots) + \ldots$$

where $\alpha \in \underline{k}$, then

$$\mathrm{ld}(f) = h_1(h_1(\ldots(h_1h))\ldots)$$

where $h \in H_q$ and

$$\text{X-length}\ (\ \mathrm{ld}\ (f)\ )\ =\ (s-2)v + q < p$$

which follows form the minimality of s among integers satisfying such an inequality. Hence, $f \neq 0$ and $S_{(s-1)} \neq \{0\}$ in L.

Therefore, S is nilpotent of class s.

Note : If $y_1 \in F_{(v)}$, $y_1 \notin F_{(v+1)}$ and $Y = \{ y_1, y_2 \}$ is linearly independent modulo $F_{(q+1)}$, q being the smallest such integer, and $\mathrm{ld}\ (y_1) = \mathrm{ld}\ (y_2)$, then

$$y_1 = (\alpha_1 h_1 + \ldots + \alpha_r h_r) + \beta h'$$

$$y_2 = \gamma\ (\alpha_1 h_1 + \ldots + \alpha_r h_r) + \delta h$$

where $h_1, \ldots, h_r \in (\ H_v \cup \ldots \cup H_{q-1}\ )$, $h_1 \in H_v$, and either $h \in H_q$ or $h' \in H_q$. If h and h' both belong to H_q, then $h \neq h'$.
We can then write

$$y_1 = \gamma y_2 + F_{(q)}$$

so that modulo $F_{(q)}$, Y is linearly dependent and $\mathrm{ld}\ (y_1 y_2)$ has X-length $v + q$.

We also note that a non-zero element f of F is zero in L if and only if $\mathrm{ld}\ (f) \in H_p$.

Corollary 2.1 : Let L, S, Y, v, w be as in the proof of the previous theorem, $v \leq w$, and $\mathrm{ld}\ (y_1) \neq \mathrm{ld}\ (y_2)$. Then $q = w$ and S is nilpotent of class s where s is the smallest integer satisfying

$$(\ s-1\ )\ v + w \geq p$$

Proof : If $h_1 = \mathrm{ld}\ (y_1) \neq \mathrm{ld}\ (y_2) = h_2$, then in the product $y_1 y_2$ there is the non-zero term $h_1 h_2$ and hence $y_1 y_2 \neq 0$. Thus Y is linearly independent modulo $F_{(w+1)}$ and $q = w$.

Example 2.1 : Let F be a free Lie algebra on $X = \{a, b, c\}$, and $L = F / F_{(7)}$. Suppose S is the subalgebra of F generated by $Y = \{y_1, y_2\}$, where

$$y_1 = a + ab + a(ab)$$
$$y_2 = a + ab + b(bc)$$

Then, $\operatorname{ld}(y_1) = a \in X$ which implies that $v = 1$. Also Y is linearly independent modulo $F_{(4)}$, hence $q = 3$. The smallest integer s such that

$$(s - 1)1 + 3 \geqslant 7$$

is $s = 5$. Therefore, S is nilpotent of class 5.

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§ 2. Subalgebras of a Free Nilpotent Lie Algebra in General

Let F be a free Lie algebra on X and $L = F / F_{(p)}$, $p \geqslant 2$. We now determine the class of nilpotency of any subalgebra S of L using the ideas developed in the previous section. Let us assume throughout this section that the rank of S is $\geqslant 2$. We first prove

Lemma 2.1 : Let S be a subalgebra of L. Then, the integer m defined by

$$m = \min \left\{ X\text{-length}(\operatorname{ld}(y)) : y \in Y \right\},$$

where Y is some generating set for S, is an invariant of the Lie algebra S in L.

Proof : Let Y' be a generating set for S different from Y stated above, and

$$m' = \min \left\{ X\text{-length}(\operatorname{ld}(y')) : y' \in Y' \right\}$$

We now prove that $m' = m$. Suppose that $m' > m$. Then an element of

S which is non-zero modulo $F_{(m+1)}$ (such an element exists in Y) cannot be expressed in terms of the generators in Y'. On the other hand if $m' < m$, then there exists an element $y' \in Y'$ such that $y' \notin F_{(m)}$ and this element cannot be expressed in terms of the elements of Y. Therefore, $m = m'$ is an invariant of the subalgebra S of L.

Lemma 2.2 : Let S be a subalgebra of L and Y be a generating set for S. Suppose

$$m = \min \left\{ \text{X-length (ld } (y) \text{) } : y \in Y \right\}$$

Then, we can choose a pair of elements $\{ y_1, y_2 \} \subseteq Y$ such that

(i) $\{ y_1, y_2 \}$ is linearly independent modulo $F_{(q+1)}$, where q is the smallest such integer, $m \leq q < p$, and if $\{ y_i, y_j \}$ is any other pair of elements of Y which is linearly independent modulo $F_{(u+1)}$, then $u \geq q$.

(ii) length (ld $(y_1 y_2)$) is minimal among the lengths of the leading terms of such products for pairs of elements of Y satisfying (i).

Furthermore, the element

$$g = \underbrace{y_1(y_1(\ldots(y_1}_{\text{z-times}} y_2))\ldots)$$

has leading term whose X-length equals $zm + q$.

Proof : We prove the lemma by choosing a pair $\{ y_1, y_2 \}$ which satisfies the conditions above. Let $y_1 \in Y$, where X-length (ld (y_1)) is m. If y_1 is the only such element, then choose y_2 such that

$$n = \text{X-length (ld } (y_2) \text{) } = \min \left\{ \text{X-length (ld } (y) \text{) } : y \in Y - \{y_1\} \right\}$$

Then, $n > m$ and $\{y_1, y_2\}$ is linearly independent modulo $F_{(n+1)}$, $q = n$. Obviously, (i) and (ii) are satisfied.

Now suppose there are more that one element, say $\{y_1, y_2, \ldots\} = \underline{Y}$ belonging to Y and satisfying

$$\text{X-length (ld } (y_i) \text{) } = m$$

for $i = 1, 2, \ldots$. For each $y_i \in \underline{Y}$, there exists elements $y \in Y$ such that $\{y_i, y\}$ is linearly independent modulo $F_{(w+1)}$, where w is the smallest such integer. Such an element y may itself belong to $\underline{Y}$. Let $\{y_j, y\}$, $y_j \in \underline{Y}$, be such that it is linearly independent modulo $F_{(q+1)}$, and that this number q is minimal for all such pairs. We now show that $\{y_j, y\}$ is a pair that satisfies the required conditions.

Suppose that there exists a pair $\{a, b\} \subseteq Y$, $a, b \notin \underline{Y}$ and $\{a, b\}$ is linearly independent modulo $F_{(w+1)}$ with $w \leq q$. Let us assume that $\{a, b\}$ satisfies (i) and (ii). Let

$$u = \text{X-length (ld (a)) } \leq \text{ X-length (ld (b)) } \leq w .$$

For any $y_i \in \underline{Y}$, take the pair $\{y_i, a\}$. Since $u > m$, this pair is linearly independent modulo $F_{(u+1)}$, where $u \leq w$. Then

$$\begin{aligned} \text{X-length (ld (} y_i a \text{)) } &= m + u \\ &< 2u \\ &\leq \text{ X-length (ld (ab)) } \end{aligned}$$

and this contradicts the assumption that the pair $\{a, b\}$ satisfies (i) and (ii). Hence, in any choice of such a pair, one element must belong to $\underline{Y}$. Since $\{y_j, y\}$ is a'minimal' pair among those , it satisfies (i) and (ii).

Note that in $\{y_j, y\}$ above, if $y \notin \underline{Y}$, then $q = \text{X-length (ld (y))}$.

Definition 2.2 : Let S be a subalgebra of $L = F / F_{(p)}$ and Y be a generating set for S. Call a pair $\{y_1, y_2\}$ as described in Lemma 2.2 a minimal pair for Y.

Note that given a set Y there may be more than one minimal pair.

Lemma 2.3 : Let S be a subalgebra of L and suppose it possesses a generating set Y. Let $\{y_1, y_2\}$ be a minimal pair in Y and suppose that $\{y_1, y_2\}$ is linearly independent modulo $L_{(q+1)}$, where q is the minimal such integer. Then the integer q is an invariant of S in L.

Proof : For a given generating set Y of S let $\{y_1, y_2\}$ and q be as described above. Let Y' be another generating set for S and $\{y_1', y_2'\}$ be a minimal pair in Y'. Suppose $\{y_1', y_2'\}$ is linearly independent modulo $L_{(q'+1)}$. We now prove that $q' = q$.

Suppose that $q' < q$. Then, there is an element $f \in S_{(2)}$ which is non-zero modulo $L_{(m+q'+1)}$ (take $f = y_1'y_2'$), i.e., $f \notin L_{(m+q'+1)}$. This element cannot be expressed in terms of the generators in Y, since any product of elements of Y to give an element of Y-length $= 2$ cannot be non-zero modulo $L_{(t+1)}$, for an arbitrary $t < q$. But $q' < q$ and this is a contradiction. If $q' > q$, then we get a similar contradiction. Hence, the integer q is independent of the choice of the generating set Y for S and it is an invariant of S.

As a corollary to Lemma 2.2, we have

Corollary 2.2 : Let m and q be the invariants of the subalgebra S of L as described above. Then, $S \subseteq F_{(m)}$ but $S \not\subseteq F_{(m+1)}$ and $S_{(2)} \subseteq F_{(m+q)}$ but $S_{(2)} \not\subseteq F_{(m+q+1)}$.

Proof : Let Y be a generating set for S. By definition of m, there exists an element y in Y such that

$$X\text{-length} (\operatorname{ld} (y)) = m$$

Hence, $y \notin F_{(m+1)}$ and $S \not\subseteq F_{(m+1)}$. But minimality of m implies that there is no element in S whose leading term has X-length $< m$. Thus $S \subseteq F_{(m)}$.

Similarly, by definition of q, there exists a pair $\{ y_1, y_2 \}$ of elements in Y such $\{ y_1, y_2 \}$ is linearly independent modulo $F_{(q+1)}$, and X-length (ld $(y_1 y_2)$) = $m + q$. Hence, $y_1 y_2 \notin F_{(m+q+1)}$ and $S_{(2)} \not\subseteq F_{(m+q+1)}$. Minimality of this pair implies that $S_{(2)} \subseteq F_{(m+q)}$.

For any subalgebra S of $L = F / F_{(p)}$, let the triple (S, m, q) denote S with its two invariants as defined above. We now state

Theorem 2.2 : Let (S, m, q) be a subalgebra of a free nilpotent Lie algebra $L = F / F_{(p)}$. Then, S is nilpotent of class s if and only if s is the smallest integer satisfying

(*) $$(s-1) m + q \geq p$$

Proof : Let s be the smallest integer satisfying (*). Suppose Y is a generating set for S having a minimal pair $\{ y_1, y_2 \} \subseteq Y$, which is linearly independent modulo $F_{(q+1)}$ and

$$X\text{-length} (\operatorname{ld} (y_1)) = m$$

The element f of $S_{(s-1)}$ defined by

$$f = y_1(y_1(\ldots(y_1 y_2))\ldots)$$

has leading term whose X-length is $(s-2) m + q < p$. Hence, $f \notin F_{(p)}$ and $S_{(s-1)} \neq \{0\}$ in L. However, an element in $S_{(s)}$ whose leading term has minimal X-length is of the form

$$f = \eta \underbrace{y_1(y_1(\ldots(y_1}_{(s-1)\text{-times}} y_2)\ldots) + \ldots$$

which satisfies

$$X\text{-length}(\,ld\,(f)\,) = (s-1)m + q \geqslant p\,.$$

Hence, $S_{(s)} = \{0\}$ in L and S is nilpotent of class s.

Conversely, assume that S is nilpotent of class s. Let Y be a generating set for S. Then, if $f \in S_{(s)}$, $f \neq 0$ in L, which implies that $f \in F_{(p)}$. Let g be the element of $S_{(s)}$ defined by

$$g = \underbrace{y_1(y_1(\ldots(y_1}_{(s-1)\text{-times}} y_2))\ldots)\ ,$$

where $\{y_1, y_2\}$ is a minimal pair for Y. Since, X-length of the leading term of $g \geqslant p$, we have

$$(s-1)m + q \geqslant p\,.$$

Now suppose that there is an integer $a < s$ satisfying (*). Then, $(s-1) > (a-1)$ and $(s-2) \geqslant (a-1)$. Thus we have

$$(\dagger) \qquad (s-2)m + q \geqslant (a-1)m + q \geqslant p$$

Let t be the element of $S_{(s-1)}$ defined by

$$t = \underbrace{y_1(y_1(\ldots(y_1}_{(s-2)\text{-times}} y_2))\ldots)$$

Then, t is an element of $S_{(s-1)}$ with a minimal leading term. But by hypothesis $S_{(s-1)} \neq \{0\}$ in L and $t \notin F_{(p)}$. Thus

$$X\text{-length}(ld\,(t)\,) = (s-2)m + q < p$$

This contradicts $(\dagger)$. Hence, s is the smallest integer which satisfies (*)

<u>Example 2.2</u> : Let F be the free Lie algebra on generators $X = \{a, b, c\}$ and $L = F / F_{(14)}$. Suppose S is the subalgebra of L generated by $Y = \{y_1, y_2, y_3\}$, where

$$y_1 = ab + a(ab) + b(b(b(ab)))$$

$$y_2 = ab + a(ab)$$

$$y_3 = c(b(ab)) + c(c(b(ab)))$$

Then, $m = 2$ and we can take $\{y_1, y_3\}$ or $\{y_2, y_3\}$ as a minimal pair. Both of these pairs are linearly independent modulo $F_{(5)}$. Hence, $q = 4$.

The smallest integer which satisfies

$$(s-1)\,2 + 4 \geq 14$$

is $s = 6$. Thus S is nilpotent of class 6 in L.

<u>Example 2.3</u> : Let F be the free Lie algebra on generators $X = \{a, b\}$ and $L = F / F_{(4)}$. Let S be the subalgebra of L generated by $Y = \{y_1, y_2\}$, where

$$y_1 = a + ab$$

$$y_2 = 2a + ab$$

Then, $m = 1$, $q = 2$ and $s = 3$. Thus S is nilpotent of class 3.

<u>Note</u> : In $L = F / F_{(p)}$, for every positive integer $s \leq p$, there is a subalgebra which is nilpotent of class s, since taking $Y = \{y_1, y_2\}$ such that

$$y_1 \in X$$

$$y_2 \in H_{p-(s-1)}$$

the element of $S_{(s-1)}$ defined by

$$g = \underbrace{y_1(y_1(\ldots(y_1}_{(s-2)\text{-times}} y_2))\ldots)$$

satisfies

$$\text{X-length}\ (\ \mathrm{ld}\ (g)\ ) = (\ s - 2\ ) + p - (\ s - 1\ ) = p - 1$$

Therefore, $S_{(s-1)} \neq \{0\}$. But the element in $S_{(s)}$ whose leading term has smallest X-length is of the form

$$t = \psi\ \underbrace{y_1(y_1(\ldots(y_1}_{(s-1)\text{-times}}y_2))\ldots) + \ldots$$

where $\psi \in \underline{k}$, and

$$\text{X-length}\ (\ \mathrm{ld}\ (t)\ ) = (\ s - 1\ ) + p - (\ s - 1) = p$$

Hence, $S_{(s)} = \{0\}$.

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§ 3. Free Nilpotent Subalgebras of a Free Nilpotent Lie Algebra

A well-known theorem of A. I. Sirsov [17] states that the subalgebras of a free Lie algebra are themselves free Lie algebras. It is easy to see that subalgebras of free abelian Lie algebras are also free abelian, since any linearly independent subset of such a Lie algebra generates a free abelian Lie algebra. However, only in very rare cases every subalgebra of a reduced free Lie algebra is itself reduced free.

In general subalgebras of a free nilpotent Lie algebra need not be free nilpotent. In this section we determine those subalgebras of a free nilpotent Lie algebra which are free nilpotent possibly of a lower class than the original Lie algebra.

A. I. Mal'cev [13] proved the following result for groups :

" Let G be a free nilpotent group of class c, and Y a subset of G with $|Y| > 1$. Then Y generates a free nilpotent subgroup of class c if and only if Y is linearly independent modulo $G_{(2)}$. "

A. W. Montowski has stated a similar result in [14] :

" Let G be a free nilpotent group of class c. Then, a subgroup H is free nilpotent of class c if and only if

$$H_{(2)} = H \cap G_{(2)}$$

or H is a cyclic group. "

A more general version appeared in [7] :

" Let G be a nilpotent group of class c. If Y is a subset of G such that $Y \subseteq G_{(w)}$ and linearly independent modulo $G_{(w+1)}$, then Y freely generates a free nilpotent group of class $[c/w] + 1$, where $[c/w]$ is the largest integer $<$ c/w. "

It is easy to see that the converse of this statement is not necessarily true. More work has been done on this subject by S. Moran in [15]. In this section we generalize the results mentioned above for the case of Lie algebras and determine those sets in a free nilpotent Lie algebra that generate a free nilpotent Lie algebra of a possibly lower class.

<u>Theorem 2.3</u> : Let $L = F / F_{(p)}$ be a free nilpotent Lie algebra of class p on free generators X' and S be a subalgebra of L. Then S is free nilpotent of class p if and only if it has a generating set which is linearly independent modulo $L_{(2)}$

<u>Proof</u> : First we note that a set generates a free nilpotent Lie algebra of class p in L if and only if every finite subset of it does the same. (see [14]). Hence, it is sufficient to consider finitely generated subalgebras only. Let

$$Y' = \{ y_1', \ldots, y_m' \}$$

be a generating set for S, where

$$y_i' = \sum \alpha_j x_j' + f_i'$$

$\alpha_j \in \underline{k}$ (the field), $x_j' \in X'$ and $f_i' \in L_{(2)}$. Then, there exists a free generating set X of F such that Y' can be transformed onto $Y = \{ y_1, \ldots y_m \}$, where

(*) $$y_i = x_i + f_i$$

$x_i \in X$, $f_i \in L_{(2)}$, for $i = 1, \ldots, m$. Let H be a Hall set on X.

Let[redacted] $\{ x_1, \ldots, x_m \} = \underline{X} \subseteq X$, [redacted]. Let A be the subalgebra of F generated by $\underline{X}$. Obviously A is a free nilpotent Lie algebra of class p. We now show that

$$S \cong A / A_{(p)}$$

Let $\sigma : A \to S$ be the map defined by

$$\sigma : x_i \mapsto y_i \quad \text{for } i = 1, \ldots, m$$

All we need to prove is that Kernel $(\sigma) = A_{(p)}$. Obviously $\sigma(A_{(p)}) \subseteq L_{(p)} = \{ 0 \}$. Thus $A_{(p)} \subseteq$ Kernel (σ). Now suppose that there exists an element $g \in A_{(w)}$, $g \notin A_{(w+1)}$, $w < p$, such that

$$\sigma(g) = 0$$

We can write

$$g = \sum_i \mu_i h_i(x_1, \ldots, x_m) + d$$

where $\mu_i \in \underline{k}$, $h_i(x_1, \ldots, x_m) \in H_w$, $d \in L_{(w+1)}$ and not all μ_i

equal to zero. Then

$$\sigma(g) = \sum \mu_i h_i(y_1, \ldots, y_m, \text{[redacted]}) + d'$$

where $d' \in L_{(w+1)}$. By (*) we have

$$\sigma(g) = \sum \mu_i h_i(x_1, \ldots, x_m) + b + d' ,$$

where $h_i(x_1, \ldots, x_m) \in H_w$ and every summand of b contains at least one expression of the form f_i as in (*). Then, $b \in L_{(w+1)}$, and $\sigma(g) = 0$ implies that $b + d' = 0$, since otherwise an element of length $\geqslant w+1$ would be expressed as a linear combination of elements of H_w. Hence we have

$$\sum_i \mu_i h_i(x_1, \ldots, x_m) = 0$$

This contradicts the linear independence of the basis elements in H, since not all μ_i is equal to zero. Thus $\sigma(g) \neq 0$, Kernel $(\sigma) = A_{(p)}$ and S is isomorphic to $A / A_{(p)}$.

Therefore S is free nilpotent of class p.

Conversely, suppose that S is a free nilpotent subalgebra of class p in L. Let Y be a set of free generators for S. If there exists an element $y \in Y$ such that $y \in L_{(2)}$, then for any y' such that $y' \in Y - \{y\}$, the element $g \in S_{(p-1)}$ defined by

$$g = \underbrace{y'(y'(\ldots(y'}_{(p-2)\text{-times}}y))\ldots)$$

belongs to $L_{(p)}$ and hence is zero, which contradicts S being free nilpotent of class p. Thus there is no $y \in Y$ such that $y \in L_{(2)}$.

Now suppose that Y is not linearly independent modulo $L_{(2)}$. Then there exists $y \in Y$ such that

$$y = \sum_{\substack{j \\ y_j \neq y}} \alpha_j y_j + L_{(2)}$$

where $\alpha_j \in \underline{k}$, $y_j \in Y$. Let us put

$$z = \sum_{\substack{j \\ y_j \neq y}} \alpha_j y_j$$

Then $yz \in L_{(3)}$ and the element $f \in S_{(p-1)}$ defined by

$$f = \underbrace{y(y(\ldots(y}_{(p-2)\text{-times}}z))\ldots)$$

belongs to $L_{(p)}$ and $f = 0$, which is a contradiction.

Therefore, Y is linearly independent modulo $L_{(2)}$.

Let F be a free Lie algebra on free generators X with a Hall basis H constructed on it.

<u>Lemma 2.4</u> : Let $D \subseteq (H_m \cup \ldots \cup H_{2m-1})$, $m \geq 1$. Then D is a free generating set in the free Lie subalgebra of F that it generates.

<u>Proof</u> : If a set generates a free Lie algebra, then every subset of it does the same. Let C_m be the free generating set for $F_{(m)}$ as described in Chapter 1. Then $D \subseteq C_m$ and thus it is a free generating set in the subalgebra that it generates.

<u>Definition 2.3</u> : Let p be a positive integer, $p > 1$, and $[p/k]$ denote the largest integer $<$ p/k, for $k \leq p$. For a given

pair of positive integers p and s , both ≥ 2, we define $K = \{k_1,\dots,k_u\}$ to be the maximal set of consecutive integers satisfying

$$(*) \qquad [p/k_1] = [p/k_2] = \dots = [p/k_u] = s-1$$

Example 2.4 : Let $p = 11$, $s = 3$. Then $K = \{4,5\}$, since these are the only integers satisfying (*).

Lemma 2.5 : For given positive integers p, $s \geq 2$, let $K = \{k_1,\dots,k_u\}$ be as defined above. Then, for any $k_i \in K$ we have

(i) $(s-1)k_i < p$ and $s\,k_i \geq p$

(ii) $k_u < 2k_1$

Proof : (i) By definition of K, we have $[p/k_i]\,k_i < p$, and $([p/k_i]+1)\,k_i \geq p$.

(ii) Assume $k_u \geq 2k_1$. Then

$$\begin{aligned}(s-1)k_u &\geq (s-1)(k_1+k_1)\\ &\geq s\,k_1\\ &\geq p\end{aligned}$$

This contradicts (i). Hence, $k_u < 2k_1$.

Lemma 2.6 : Let p, s and K be as defined above. Then, for any positive integer $w < s$, we have

$$(w+1)k_1 > w\,k_u$$

Proof : Suppose for some positive integer $w < s$ we have

$$(w+1)k_1 \leq w\,k_u$$

Then

$$\begin{aligned}&(w+1)k_1 + (s-(w+1))k_1\\ &\leq w\,k_u + (s-(w+1))k_1\end{aligned}$$

$$\leq \quad w\,k_u + (s - (w+1))\,k_u$$
$$= \quad (s-1)\,k_u$$
$$< \quad p\,.$$

But $(w+1)\,k_1 + (s-(w+1))\,k_1 = s\,k_1 \geq p$, which contradicts the inequality obtained above.

Therefore, $(w+1)\,k_1 > w\,k_u$.

Lemma 2.7 : Let $L = F / F_{(p)}$ and $K = \{ k_1, \ldots, k_u \}$ be as defined above for a given positive integer s. Then the set

$$D = (H_{k_1} \cup \ldots \cup H_{k_u})$$

freely generates a free nilpotent Lie algebra of class s in L.

Proof : Since $k_u < 2k_1$, D is a set of free generators in the subalgebra N of F that it generates. (by Lemma 2.4). All we need to show is that $N_{(s)} = \{0\}$ in L and if $f \in N_{(w)}$, $f \notin N_{(w+1)}$, $w < s$, then $f \neq 0$ in L.

Let $D = \{ d_1, d_2, \ldots \}$. In $N_{(s)}$ an element whose leading term has minimal X-length is of the form

$$g = \alpha\, d_{i_1} (d_{i_2} (\ldots (d_{i_{s-1}} d_{i_s})) \ldots) + \ldots$$

where $\alpha \in \underline{k}$, $d_{i_j} \in H_{k_1}$ and

$$\text{X-length} (\text{ld}\, (g)) = s\,k_1 \geq p\,.$$

Hence, $g = 0$ and $N_{(s)} = \{0\}$ in L.

Since $D \subseteq H$, we can give it an order which coincides with that of H. Furthermore, since $(a+1)\,k_1 < a\,k_u$ for any positive integer $a < s$, if $\underline{H}$ is a Hall set constructed on D, then

$$(\underline{H}_1 \cup \ldots \cup \underline{H}_{s-1})$$

is a subset of H. (We give this set an order which coincides with that of H). (See note at the end of this chapter)

Suppose now that $f \in N_{(w)}$, $f \notin N_{(w+1)}$, where $w < s$. Then

$$f = \sum \alpha_j h_j(d_1, \ldots) + h$$

where $\alpha_j \in \underline{k}$, $h_j \in \underline{H}_w$ and $h \in N_{(w+1)}$. If $f = 0$, then

$$\sum \alpha_j h_j(d_1, \ldots) = 0$$

and all h_j are non-zero in L. This contradicts the linear independence of the basis elements of L, since $\underline{H}_w \subseteq H$. Hence $f \neq 0$, and D freely generates a free nilpotent Lie algebra of class s in L.

<u>Theorem 2.4</u> : Let $L = F / F_{(p)}$ be a free nilpotent Lie algebra of class p and s, $K = \{ k_1, \ldots, k_u \}$ be as defined above. If S is a subalgebra of L which possesses a generating set $Y \subseteq L_{(k_1)}$ and linearly independent modulo $L_{(k_u+1)}$, then S is free nilpotent of class s and Y is a set of free generators for S. The converse of this statement is not generally [redacted] true.

<u>Proof</u> : Let D and N be as in the proof of the previous lemma. As mentioned before we can take S to be finitely generated witout loss of generality. Let $Y = \{ y_1, \ldots, y_m \}$ be a generating set for S. Then by hypothesis we can express the elements of Y as

$$y_i = \sum \alpha_j h_j + f_i$$

where $\alpha_j \in \underline{k}$, $h_j \in D$ and $f_i \in L_{(k_u+1)}$. Put

$$a_i = \sum \alpha_j h_j$$

for $i = 1, \ldots, m$ so that

(*) $$y_i = a_i + f_i \, .$$

Then the set $\{a_1, \ldots, a_m\} \subseteq N$ and it is linearly independent modulo $N_{(2)}$. By Theorem 2.3, $\{a_1, \ldots, a_m\}$ freely generates a free nilpotent subalgebras of class s in L. Let $A / A_{(s)}$ be the free nilpotent subalgebra of L generated by $\{a_1, \ldots, a_m\}$. All we need to show is that

$$S \cong A / A_{(s)}$$

in L. Let $\sigma : A \to S$ be the map defined by

$$\sigma : a_i \mapsto y_i$$

for $i = 1, \ldots, m$. If $f \in A_{(s)}$, then obviously $\sigma(f) = 0$ in L. Hence, $\sigma(A_{(s)}) = \{0\}$ and

$$A_{(s)} \subseteq \text{Kernel}(\sigma)$$

Now suppose there exists an element $f \in A_{(w)}$, $f \notin A_{(w+1)}$, $w < s$, such that $\sigma(f) = 0$. We can express f as

$$f = \sum \gamma_j h_j(a_1, \ldots, a_m) + g$$

where $\gamma_j \in \underline{k}$, $h_j \in \underline{H}_w$ ($\underline{H}$ being a Hall set constructed on $\{a_1, \ldots a_m\}$) and $g \in A_{(w+1)}$. Then

$$\begin{aligned} \sigma(f) &= \sum \gamma_j \sigma(h_j(a_1, \ldots, a_m)) + \sigma(g) \\ &= \sum \gamma_j h_j(y_1, \ldots, y_m) + \sigma(g) \\ (\dagger) \qquad &= \sum \gamma_j h_j(a_1, \ldots, a_m) + t + \sigma(g) \end{aligned}$$

where t contains summands each of which has at least one element of the form f_i as in (*), $\sigma(g) \in L_{(w+1)}$. But by Lemma 2.6

$$(w+1)k_1 > w k_u$$

and thus we have

$$\text{X-length}(\,\mathrm{ld}(\sigma(g))\,) > \text{X-length}(\,\mathrm{ld}(h_j(a_1,\dots,a_m))\,)$$

Let $\{h_1,\dots,h_r\}$ be those h_j used in (†) whose distinct leading terms have minimal X-length among leading terms of all the h_j. Then every other term in (†) has leading term with greater X-lengths than those of $\{h_1,\dots,h_r\}$. Thus $\sigma(f) = 0$ implies that

$$\sum_{i=1}^{r} \gamma_i \; (h_i(a_1,\dots,a_m)) = 0$$

This contradicts the linear independence of the basis elements of $A/A_{(s)}$. Hence $\sigma(f) \neq 0$ and Kernel $(\sigma) = A_{(s)}$.

Therefore, $S \cong A/A_{(s)}$ and S is free nilpotent of class s in L.

The example which follows shows that the converse of this theorem is not necessarily true.

<u>Example 3.5</u> : Let $L = F/F_{(9)}$, where F is the free Lie algebra on $X = \{a, b, c\}$. Let $s = 3$. Then $K = \{3, 4\}$. Let S be the subalgebra of L generated by $Y = \{y_1, y_2\}$, where

$$y_1 = (ab)$$
$$y_2 = b(b(b(ac)))$$

Then $y_1 \in H_2$ and $y_2 \in H_5$, where H is the Hall basis for F. Thus $Y \subseteq F_{(k_1)} = F_{(3)}$ and Y is not linearly independent modulo $F_{(k_u+1)} = F_{(3)}$, but S is free nilpotent of class 3 in L since $S_{(3)} = \{0\}$ and

$$\text{X-length} (y_1 y_2) = 7$$

implies that $y_1 y_2 \neq 0$.

Definition 2.4 : Let p, s, $K = \{k_1, \ldots, k_u\}$ be as defined before and assume that $s > 2$. For a given positive integer $r \geq 1$, $r < k_1$ we define a maximal set of consecutive integers $T_r = \{t_1, \ldots, t_n\}$ as follows :

An integer $t \in T_r$ if and only if it satisfies the following inequalities :

(1) $$(s-1)(k_1 - r) + t \geq p$$

(2) $$(s-2)t + (k_1 - r) < p$$

The set $T_r = T_r(p,s,r)$, determined by r, p and s, is empty for certain r, p and s but it is non-empty for small enough r.

Example 2.6 : Take $p = 19$, $s = 3$, $r = 1$. Then we have $K = \{7,8,9\}$ and $T_1 = \{7,8,9,10,11,12\}$. Similarly

$$T_2 = \{9,10,11,12,13\}$$
$$T_3 = \{11,12,13,14\}$$
$$T_4 = \{13,14,15\}$$
$$T_5 = \{15,16\}$$
$$T_6 = \{17\}$$

and for $r \geq 7$ (when p and s is fixed) $T_r = \phi$.

Lemma 2.8 : Let $K = \{k_1, \ldots, k_u\}$ and $T_r = T_r(p,s,r)$

be as defined above. Let $T = \{t_1, \dots, t_i\}$ be a subset of T_r such that $t_1, \dots, t_i \leq k_u$. Then

(i) $(k_1 - r) + t_1 > t_i$

(ii) $k_1 \leq t_1$

Proof : (i) Assume that $(k_1 - r) + t_1 \leq t_i$. Then

$$\begin{aligned}(s-1)(k_1 - r) + (s-1)t_1 &\leq (s-1)t_i \\ &\leq (s-1)k_u \\ &< p\end{aligned}$$

Since $s > 2$, this would imply that

$$(s-1)(k_1 - r) + t_1 < p$$

which contradicts (1). Therefore $(k_1 - r) + t_1 > t_i$.

(ii) Assume that $t_1 < k_1$. By (1) we have

$$(*) \qquad (s-1)(k_1 - r) + t_1 \geq p$$

If $t_1 \leq (k_1 - r)$ then by the above inequality we have

$$\begin{aligned}&(s-1)(k_1 - r) + (k_1 - r) \\ \geq\ &(s-1)(k_1 - r) + t_1 \\ \geq\ &p\end{aligned}$$

and thus

$$s(k_1 - r) \geq p .$$

But since $k_1 - r < k_1$ we can write

$$(s-1)(k_1 - r) < p .$$

This contradicts the fact that k_1 is tha smallest positive integer satisfying two inequlaties of the form (1) and (2). Hence, $t_1 > k_1 - r$.

Now using (*) we have

$$s\,t_1 = (s-1)\,t_1 + t_1$$
$$> (s-1)(k_1 - r) + t_1$$
$$\geqslant p$$

Hence $t_1 < k_1$ satisfies

$$(s-1)\,t_1 < p$$
$$s\,t_1 \geqslant p$$

which again contradicts k_1 being the smallest positive integer satisfying two such inequalities.

Therefore $t_1 \geqslant k_1$.

<u>Lemma 2.9</u> : Let K', T_r, $T = \{t_1,\ldots,t_i\}$ be as in the previous lemma. Then for any positive integer $c < s$ we have

$$c\,(k_1 - r) + t_1 > c\,t_i$$

<u>Proof</u> : Suppose that $c(k_1 - r) + t_1 \leqslant ct_i$. Then

$$c\,(k_1 - r) + ((s-1) - c)(k_1 - r) + t_1$$
$$\leqslant c\,t_i + ((s-1) - c)(k_1 - r).$$

Then,

$$\text{LHS} = (s-1)(k_1 - r) + t_1$$
$$\leqslant c\,t_i + ((s-1) - c)\,t_i$$
$$= (s-1)\,t_i$$
$$\leqslant (s-1)\,k_u$$
$$< p$$

But $\text{LHS} = (s-1)(k_1 - r) + t_1 < p$ and this contradicts (1).

Therefore, $c(k_1 - r) + t_1 > ct_i$ for any positive integer $c < s$.

<u>Lemma 2.10</u> : Let $L = F / F_{(p)}$, s, K, T_r, and $T = \{t_1, \dots, t_i\}$ be as defined before. Suppose $y \in H_{k_1-r}$ and let

$$D = \{y\} \cup (H_{t_1} \cup \dots \cup H_{t_i})$$

Then, D freely generates a free nilpotent Lie subalgebra of class s in L.

<u>Proof</u> : $D \subseteq H$ and hence it is linearly independent. Furthermore length $(y) = k_1 - r$ and by Lemma 2.9

$$(*) \qquad c (k_1 - r) + t_1 > c\, t_i$$

Suppose that D is given the same order as H and let $\underline{H}$ be a Hall set constructed on D. If $c < s-1$, in $\underline{H}_{c+1}$ an element f which has a minimal X-length satisfy

$$\text{X-length } (f) = c(k_1 - r) + t_1 \,.$$

In $\underline{H}_c$, if g is an element with a maximal X-length, then

$$\text{X-length } (g) = c t_i$$

Then the inequality (*) implies that by a suitable choice of a D-length preserving ordering in $\underline{H}_1 \cup \dots \cup \underline{H}_{s-1}$ we also preserve X-lengths. Then, $\underline{H}_1 \cup \dots \cup \underline{H}_{s-1}$ becomes a subset of H, which implies that elements of this set are linearly independent in F. All we need to show is that elements of $\underline{H}_1 \cup \dots \cup \underline{H}_{s-1}$ are non-zero in L, and if N is the subalgebra of F generated by D then $N_{(s)} = \{0\}$ in L.

In $\underline{H}_{s-1}$, an element f with a maximal X-length satisfies

$$\begin{aligned} \text{X-length } (f) &= (s - 1) t_i \\ &\leq (s - 1) k_u \\ &< p \end{aligned}$$

Thus the elements of $\underline{H}_{s-1}$ are non-zero in L. Since elements of $\underline{H}_1 \cup \ldots \cup \underline{H}_{s-2}$ have smaller X-lengths, they are also non-zero in L.

In $N_{(s)}$, an element g whose leading term has a minimal X-length is of the form

$$g = \zeta\, y(y(\ldots(yd))\ldots) + \ldots$$

where $\zeta \in \underline{k}$, $d \in H_{t_1}$ and $y \in H_{k_1-r}$. Then

$$\text{X-length } (\mathrm{ld}(g)) = (s-1)(k_1 - r) + t_1 \geq p.$$

by (1). Hence, $g \in F_{(p)}$ and it is zero in L. Thus $N_{(s)} = \{0\}$ in L.

Therefore, D freely generates a free nilpotent Lie algebra of class s in L.

<u>Theorem 2.5</u> : Let $L = F / F_{(p)}$ be a free nilpotent Lie algebra and $K = \{k_1, \ldots, k_u\}$, s, T_r, $T = \{t_1, \ldots, t_i\}$ be as defined before. Then,

(i) If S is a subalgebra of L which possesses a set of generators Y, where only one element $y \in Y$ belongs to $L_{(k_1-r)}$ and non-zero modulo $L_{(k_1-r+1)}$, and $Y - \{y\} \subseteq L_{(t_1)}$ and linearly independent modulo $L_{(t_i+1)}$, then S is free nilpotent of class s in L and Y is a set of free generators in S.

(ii) If S is a two-generator subalgebra of L on $Y = \{y_1, y_2\}$ such that $y_1 \in L_{(k_1-r)}$, $y_1 \notin L_{(k_1-r+1)}$ and $y_2 \in L_{(t_1)}$, $y_2 \notin L_{(t_n+1)}$, then S is free nilpotent of class s and Y is a set of free generators for S.

<u>Proof</u> : (i) Since y is the only element with the property

that $y \in L_{(t_1)}$ (by Lemma 2.8) we are only interested in the X-length of the leading term of y. Hence we can take $y \in H_{k_1-r}$ without loss of generality. Let

$$D = \{y\} \cup (H_{t_1} \cup \ldots \cup H_{t_i})$$

and N be the free subalgebra of F generated by D. As in the previous cases it is sufficient to consider finitely generated subalgebras only. Let S have the generating set $Y = \{y, y_1, \ldots, y_m\}$, where

$$y_i = \sum \alpha_j d_j + f_i$$

for $i = 1, \ldots, m$, $\alpha_j \in \underline{k}$, $d_j \in D - \{y\}$, $f_i \in L_{(t_i+1)}$. Put

$$a_i = \sum \alpha_j d_j$$

for $i = 1, \ldots, m$ so that

(*) $$y_i = a_i + f_i .$$

Then the set $\{y, a_1, \ldots, a_i\} \subseteq N$ and it is linearly independent modulo $N_{(2)}$. Hence it freely generates a free nilpotent subalgebra of class s in L by Theorem 2.3. Let A be the subalgebra of F generated by $\{y, a_1, \ldots, a_m\}$. All we need to show is that

$$S \cong A / A_{(s)}$$

Let $\sigma : A \to S$ be the map defined by

$$\sigma : a_i \mapsto y_i \quad \text{for } i = 1, \ldots, m$$
$$y \mapsto y$$

Obviously σ can be extended to A. If $f \in A_{(s)}$, then $\sigma(f) = 0$.

Hence $\sigma(A_{(s)}) = \{0\}$ and

$$A_{(s)} \subseteq \text{Kernel}(\sigma) .$$

Now assume that there exists an element $f \in A_{(w)}$, $f \notin A_{(w+1)}$, $w < s$, such that $\sigma(f) = 0$. Let $\underline{H}_1 \cup \ldots \cup \underline{H}_{s-1}$ be a Hall basis for $A / A_{(s)}$ constructed on $\{y, a_1, \ldots, a_m\}$. Then we can write

$$f = \sum_j \xi_j h_j(y, a_1, \ldots, a_m) + g$$

where $\xi_j \in \underline{k}$, $h_j \in \underline{H}_w$, $g \in A_{(w+1)}$ and not all ξ_j equal to zero. Also since $w < s$, $h_j \neq 0$ in L. Then

$$\sigma(f) = \sum \xi_j h_j(y, y_1, \ldots, y_m) + \sigma(g)$$

$$(\dagger) \qquad = \sum \xi_j h_j(y, a_1, \ldots, a_m) + b + \sigma(g) ,$$

where b contains summands each of which has at least one element of the form f_i as in (*), and $\sigma(g) \in L_{(w(k_1-r)+t_1)}$ since $A_{(w+1)} \subseteq L_{(w(k_1-r)+t_1)}$. But by Lemma 2.9

$$w(k_1 - r) + t_1 > w\, t_i$$

Hence, we have

$$\text{X-length}(\, \text{ld}(\sigma(g))\,) > \text{X-length}(\, \text{ld}(h_j(y, a_1, \ldots, a_m))\,)$$

for all h_j used in $(\dagger)$.

Let $\{h_1, \ldots, h_r\}$ be those h_j occuring in $(\dagger)$ whose leading terms have minimal X-length among leading terms of such h_j. Then every other term in the sum $(\dagger)$ has leading term with greater X-length. Thus, $\sigma(f) = 0$ would imply that

$$\sum_{j=1}^{r} \xi_i \; (h_i(y, a_1, \ldots, a_m)) = 0$$

This contradicts the linear independence of basis elements of $A / A_{(s)}$.
Hence, $\sigma(f) \neq 0$ and Kernel $(\sigma) = A_{(s)}$.

Therefore, $S \cong A / A_{(s)}$ and S is free nilpotent of class s in L.

(ii) Take $y_1 \in H_{k_1 - r}$ and let

$$y_2 = \sum \alpha_j h_j + g$$

where $\alpha_j \in \underline{k}$, $h_j \in (H_{t_1} \cup \ldots \cup H_{t_n})$, $g \in L_{(t_n+1)}$. (Note that $T = \{t_1, \ldots, t_n\}$) Obviously Y generates a free Lie subalgebra of F. If $f \in S_{(s)}$, then

$$X\text{-length} (\text{ld} (f)) \geqslant (s-1)(k_1 - r) + t_1 \geqslant p .$$

Hence $S_{(s)} = \{ 0 \}$ in L. On the other hand if $e \in S$ such that $e \notin S_{(s)}$, then

$$X\text{-length} (\text{ld} (e)) \leqslant (s-2)t_n + (k_1 - r) < p$$

so that $e \neq 0$ in L.

Therefore, S is free nilpotent of class s in L with free generating set Y.

In Definition 2.4 we assumed that $s > 2$. In fact, if $s = 2$, then the inequalities (1) and (2) would require T_r to be an infinite set. The case $s = 2$ corresponds to free abelian subalgebras. In this case the sets $K = \{ k_1, \ldots, k_u \}$ satisfy

$$k_1 = [p/2] + 1$$

$$k_2 = [p/2] + 2$$

$$\vdots$$

$$k_u = p - 1 = [p/2] + u \quad .$$

For a given $r \geq 1$, we can now define the sets T_r as follows :

An integer $t \geq 1$ belongs to T_r if and only if it satisfies the following inequalities :

(1)' $\qquad (k_1 - r) + t \geq p$

(2)' $\qquad t < p$

Let $T_r = \{t_1, \ldots, t_n\}$. Obviously if t_1 satisfies (1)' above, then any $t \geq t_1$, $t < p$ also satisfies (1)'. Hence, $t_n = k_u = p-1$ and for every value of r, $1 \leq r < k_1$, the sets T_r are non-empty. Furthermore the sets $T = \{t_1, \ldots, t_i\}$ (as defined in Lemma 2.8) coincide with T_r for every r.

Let $1 \leq r < k_1 = [p/2] + 1$. Then, in the set $T_r = \{t_1, \ldots, t_n\}$ we can obtain the value of t_1 as follows :

We want $(k_1 - r) + t_1 = p$. But $k_1 = [p/2] + 1$. Hence

$$([p/2] + 1) - r + t_1 = p$$

$$t_1 = p - [p/2] - 1 + r$$

But we have

$$p - [p/2] = [p/2] + 1 \quad \text{if } p \text{ is odd}$$
$$= [p/2] + 2 \quad \text{if } p \text{ is even}$$

Thus

$$t_1 = [p/2] + r \ \underset{\uparrow \text{if } p \text{ is even}}{(+1)}$$

Then, we can list

$$t_2 = [p/2] + (r+1) \ \underset{\uparrow \text{if } p \text{ is even}}{(+1)}$$

$$\vdots$$

$$t_n = p - 1$$

Thus for the case $s = 2$, the elements of the sets K and T_r can be written explicitly in terms of p and r. We now state

<u>Theorem 2.6</u> : Let $L = F / F_{(p)}$ be a free nilpotent Lie algebra and S be a subalgebra of L.

If S possesses a set of generators Y such that only one element $y \in Y$ belongs to $L_{([p/2]+1-r)}$, $y \notin L_{([p/2]+2-r)}$ and $Y - \{y\} \subseteq L_{([p/2]+r)}$ and linearly independent in L, then S is free abelian. (+1 if p is even)

<u>Proof</u> : In $S_{(2)}$ an element whose leading term has minimal X-length is of the form

$$g = \chi\, yy_i + \dots$$

where $\chi \in \underline{k}$, and $y_i \in Y - \{y\}$ so that $\mathrm{ld}\,(y_i) \in H_{[p/2]+r}$. Suppose p is odd Then

$$\begin{aligned} X\text{-length}\ (\ \mathrm{ld}\ (g)\ ) &= (\ [p/2] + 1 - r\ ) + (\ [p/2] + r\ ) \\ &= [p/2] + [p/2] + 1 \\ &= p \end{aligned}$$

Hence $g = 0$ in L and $S_{(2)} = \{0\}$ in L. If p is even, the result follows similarly

Therefore, S is free abelian

<u>Example 2.7</u> : Let F be a free Lie algebra on X and $L = F / F_{(9)}$. If $s = 2$, then

$$K = \{5,6,7,8\}$$

and for r satisfying $1 \leq r < 5$, we have

$$T_1 = \{5,6,7,8\}$$
$$T_2 = \{6,7,8\}$$
$$T_3 = \{7,8\}$$
$$T_4 = \{8\}$$

By determining free abelian subalgebras of a free nilpotent Lie algebra we also determine all abelian subalgebras as the following theorem states :

Theorem 2.7 : Let $L = F / F_{(p)}$ and S be a subalgebra of L. If S is abelian, then S is free abelian.

Proof : Assume that there exists an abelian subalgebra S of L such that S is not free abelian. Let Y be a minimal generating set for S. Since S is not free abelian, for some subset $\{y_1, y_2, \ldots, y_n\}$ of Y we have

$$(*) \qquad \sum_{i=1}^{n} \alpha_i y_i = 0$$

where $\alpha_i \in \underline{k}$ and not all α_i equal to zero. Then

$$y_n = \sum_{i=1}^{n-1} \alpha_i \alpha_n^{-1} y_i$$

which implies that $Y - \{y_n\}$ also generates S. This contradicts the minimality of Y as a generating set. Hence a relation of the form $(*)$ cannot exist.

Therefore S is free abelian.

Definition 2.5 : Let p , s , $K = \{k_1, \ldots, k_u\}$ be as defined before and assume that $s > 2$. For a given integer $b \geq 1$, $b < p-k_u$, we define a maximal set of consecutive integers $W_b = \{w_1, \ldots, w_e\}$ as follows :

An integer $w \geq 1$ belongs to W_b if and only if it satisfies

(3) $$(s - 1) w + (k_u + b) \geq p$$

(4) $$(s - 2) (k_u + b) + w < p$$

For certain integers p , s , b the set W_b is empty but for given p and s and small enough b it is non-empty.

Example 2.8 : Let $p = 19$, $s = 3$. Then $K = \{7,8,9\}$. For given $b \geq 1$ we have

$$W_1 = \{5,6,7,8\}$$
$$W_2 = \{4,5,6,7\}$$
$$W_3 = \{4,5,6\}$$
$$W_4 = \{3,4,5\}$$
$$W_5 = \{3,4\}$$
$$W_6 = \{2,3\}$$
$$W_7 = \{1\}$$
$$W_8 = \{1\}$$

and $W_b = \phi$ for $b \geq 9$.

Lemma 2.11 : Let p , s , $K = \{k_1, \ldots, k_u\}$, W_b be as above. Suppose $W = \{w_j, \ldots, w_e\}$ is a subset of W_b satisfying

$$w_j, \ldots, w_e \geq k_1 .$$

Then, $(k_u + b) < 2w_j$.

Proof : By (4) we have

(*) $$(s - 2) (k_u + b) + w_j < p .$$

Since $k_1 \leq w_j$ we also have

$$(s - 2) (k_u + b) + k_1 < p .$$

Assume that $(k_u + b) \geq 2w_j \geq 2k_1$. Then

$$(s-2)(k_u+b) + k_1 \geq (s-2)(k_1+k_1) + k_1$$

$$\geq sk_1$$

$$\geq p .$$

This contradicts the inequality obtained above.

Therefore, $(k_u+b) < 2w_j$.

<u>Lemma 2.12</u> : Let $W_b = \{w_1,\ldots,w_e\}$. Then, $w_e \leq k_u$.

<u>Proof</u> : Suppose that $w_e > k_u$. Then by (4) we have

(†) $$(s-2)(k_u+b) + w_e < p ,$$

where $b \geq 1$. If $k_u + b > w_e$, then

$$(s-2)w_e + w_e = (s-1)w_e < p$$

and since $w_e > k_u$, $sw_e \geq p$. This contradicts k_u being the largest integer satisfying

$$s\,k_u \geq p$$

$$(s-1)k_u < p .$$

On the other hand if $k_u + b < w_e$, then (†) implies that

$$(s-1)(k_u+b) < p$$

and we also have $s(k_u+b) > s\,k_u \geq p$ which leads to the same contradiction.

Therefore, $w_e \leq k_u$.

<u>Lemma 2.13</u> : Let K, W_b, $W = \{w_j,\ldots,w_e\}$ be as defined before. Then for any positive integer $c < s$ we have

$$(c+1)w_j > w_e + (c-1)(k_u+b)$$

<u>Proof</u> : Suppose for some c, $0 < c < s$, we have

$$w_e + (c-1)(k_u+b) \geq (c+1)\, w_j$$

Then

$$\begin{aligned} w_e + (c-1)(k_u+b) &+ ((s-2)-(c-1))(k_u+b) \\ &\geq (c+1)\, w_j + ((s-2)-(c-1))(k_u+b) \\ &\geq (c+1)\, w_j + ((s-2)-(c-1))\, w_j \\ &= s\, w_j \\ &\geq p\,. \end{aligned}$$

But the left hand side of the above inequality is equal to $w_e + (s-2)(k_u+b)$ which is less than p by (4). This is a contradiction. Therefore we have

$$w_e + (c-1)(k_u+b) < (c+1)\, w_j$$

for all $c < s$, $c > 0$.

<u>Lemma 2.14</u> : Let $L = F / F_{(p)}$ and s, K, W_b, W be as defined before. Suppose $y \in H_{k_u+b}$ and

$$D = (H_{w_j} \cup \ldots \cup H_{w_e}) \cup \{y\}$$

Then, D freely generates a free nilpotent Lie algebra of class s in L.

<u>Proof</u> : D is a subset of H and hence it is linearly independent. Furthermore, length $(y) = k_u+b$ and by the previous lemma for any positive integer $c < s$ we have

$$(*) \qquad (c+1)\, w_j > w_e + (c-1)(k_u+b)$$

Suppose D is given the same order as H and let $\underline{H}$ be a Hall set constructed on D. If $c < s-1$, in $\underline{H}_{c+1}$ an element f which has a minimal X-length satisfies

$$\text{X-length } (f) = (c+1)\, w_j$$

In $\underline{H}_c$, if g is an element with a maximal X-length, then

$$\text{X-length } (g) = (c-1)(k_u+b) + w_e$$

Hence, due to (*), by a suitable choice of a D-length preserving order in $(\underline{H}_1 \cup \ldots \cup \underline{H}_{s-1})$ we also preserve X-lengths and thus $\underline{H}_1 \cup \ldots \cup \underline{H}_{s-1}$ becomes a subset of H. Furthermore, in $\underline{H}_{s-1}$, an element t with a maximal X-length satisfies

$$\text{X-length } (t) = (s-2)(k_u+b) + w_e$$
$$< p \,.$$

Hence the elements of $(\underline{H}_1 \cup \ldots \cup \underline{H}_{s-1})$ are non-zero in L.

Now let N be the subalgebra of F generated by D. All we need to show is that $N_{(s)} = \{0\}$ in L.

In $N_{(s)}$, an element whose leading term (with respect to the basis $H_1 \cup \ldots \cup H_{p-1}$) has a minimal X-length is of the form

$$g = \alpha\, h_{i_1}(h_{i_2}(\ldots(h_{i_{s-1}} h_{i_s}) + \ldots$$

where $\alpha \in \underline{k}$, $h_{i_k} \in \underline{H}_{w_j}$. Then

$$\text{X-length } (\,\text{ld } (g)\,) = s\, w_j$$
$$\geq p \,.$$

Thus, $N_{(s)} = \{0\}$ and the set D freely generates a free nilpotent subalgebra of class s in L.

<u>Theorem 2.8</u> : Let $L = F / F_{(p)}$ and $s, K, W_b, W = \{w_j, \ldots, w_e\}$ be as defined before, for $b \geq 1$. Then,

(i) If S is a subalgebra of L which possesses a generating set Y such that only one element $y \in Y$ belongs to $L_{(k_u+b)}$, $y \notin L_{(k_u+b+1)}$ and $Y - \{y\} \subseteq L_{(w_j)}$ and linearly independent modulo $L_{(w_e+1)}$, then S is free nilpotent of class s and Y is a set of free generators for S.

(ii) If S is a two-generator subalgebra of L on generators $Y = \{y_1, y_2\}$ such that $y_1 \in L_{(w_1)}$, $y_1 \notin L_{(w_e+1)}$ and $y_2 \in L_{(k_u+b)}$ $y_2 \notin L_{(k_u+b+1)}$, then S is a free nilpotent subalgebra of class s with free generating set Y.

<u>Proof</u> : (i) Since y is the only element in Y with the property that $y \in L_{(k_u+b)}$, and we are only interested in the leading term of y, we can take $y \in H_{k_u+b}$ without loss of generality. Let D and N be as in the proof of the previous lemma. As we have mentioned before it is sufficient to consider finitely generated subalgebras only. Take $Y = \{y, y_1, \ldots, y_m\}$ to be a generating set for S, where

$$y_i = \sum_j \alpha_{ij} d_j + f_i$$

for $i = 1, \ldots, m$, $\alpha_{ij} \in \underline{k}$, $d_j \in D - \{y\}$, $f_i \in L_{(w_e+1)}$. Put

$$\sum \alpha_{ij} d_j = a_i$$

for each $i = 1, \ldots, m$ so that

$$(*) \qquad y_i = a_i + f_i$$

Then the set $\{y, a_1, \ldots, a_m\}$ is a subset of N and it is linearly independent modulo $N_{(2)}$. By Theorem 2.3 it generates a free nilpotent subalgebra of class s in L. Let A be the free subalgebra of F generated by $\{y, a_1, \ldots, a_m\}$. All we need to show is that $S \cong A / A_{(s)}$.

Let $\sigma : A \to S$ be the map defined by

$$\sigma : y \mapsto y$$
$$a_i \mapsto y_i$$

for $i = 1, \ldots, m$. Obviously σ can be extended to A. If $f \in A_{(s)}$, then $\sigma(f) \in F_{(p)}$. Hence $\sigma(A_{(s)}) = \{0\}$ in L and

$$A_{(s)} \subseteq \text{Kernel}(\sigma) .$$

Now assume that there exists an element $f \in A_{(v)}$, $f \notin A_{(v+1)}$, $v < s$, such that $\sigma(f) = 0$. Let $\underline{H}_1 \cup \ldots \cup \underline{H}_{s-1}$ be a Hall basis for $A / A_{(s)}$ constructed on $\{y, a_1, \ldots, a_m\}$. Then we can express

$$f = \sum \gamma_j h_j(y, a_1, \ldots, a_m) + g ,$$

where $\gamma_j \in \underline{k}$, $h_j \in \underline{H}_v$, $g \in A_{(v+1)}$ and not all γ_j equal to zero. Since $v < s$, $h_j \neq 0$ in L. Then

$$\sigma(f) = \sum \gamma_j h_j(y, y_1, \ldots, y_m) + \sigma(g)$$

$$(\dagger) \qquad = \sum \gamma_j h_j(y, a_1, \ldots, a_m) + a + \sigma(g)$$

where a contains summands each of which has at least one element of the form f_i as is $(*)$ and $\sigma(g) \in L_{(w_j(v+1))}$. But

$$(v+1)\, w_j > w_e + (v-1)(k_u + b)$$

(by Lemma 2.13) implies that

$$X\text{-length}(\text{ld}(\sigma(g))) > X\text{-length}(\text{ld}(h_j(y, a_1, \ldots, a_m)))$$

for all h_j used in the sum above.

Let $\{h_1,\ldots,h_r\}$ be those h_j occuring in $(\dagger)$ whose leading terms have a minimal X-length. Then every other term in $(\dagger)$ has leading term with greater X-length and $\sigma(f) = 0$ implies that

$$\sum_{j=1}^{r} \gamma_j h_j(y,a_1,\ldots,a_m) = 0$$

This contradicts the linear independence of the basis elements of $A / A_{(s)}$. Hence $\sigma(f) \neq 0$ and Kernel $(\sigma) = A_{(s)}$.

Therefore $S \cong A / A_{(s)}$ and S is free nilpotent of class s in L.

(ii) Take $y_2 \in H_{k_u+b}$ and let

$$y_1 = \sum \alpha_j h_j + g$$

where $\alpha_j \in \underline{k}$, $h_j \in (H_{w_1} \cup \ldots \cup H_{w_e})$ and $g \in L_{(w_e+1)}$. Obviously $Y = \{y_1,y_2\}$ generates a free Lie subalgebra of F, call it S. If $f \in S_{(s)}$, then

$$\begin{aligned} \text{X-length (ld } (f)\text{)} &\geq (s-1)\, w_1 + (k_u+b) \\ &\geq p\,. \end{aligned}$$

Hence, $S_{(s)} = \{0\}$ in L. If on the other hand $g \in S_{(v)}$, $v < s$, and $g \notin S_{(s)}$, then

$$\begin{aligned} \text{X-length (ld } (g)\text{)} &\leq (s-2)\,(k_u+b) + w_e \\ &< p\,. \end{aligned}$$

Therefore S is free nilpotent of class s in L with a free generating set Y.

In Definition 2.5 we assumed that $s > 2$. If $s = 2$, then in $K = \{k_1,\ldots,k_u\}$, $k_u = p-1$. This would mean that $k_u+b \geq p$ for $b \geq 1$, which leads to a trivial case.

In the cases we considered in which $|Y| > 2$, there was either one element in Y whose leading term had X-length $< k_1$ or only one element whose leading term had X-length $> k_u$. There are, however, free nilpotent subalgebras S of class s in L with generating sets Y containing two "exceptional" elements ; one whose leading term has X-length $< k_1$ and one whose leading term has X-length $> k_u$ at the same time, and $|Y| > 2$. We now look into this case in detail.

Let p, s, $K = \{k_1,\ldots,k_u\}$ be as defined before. Let $r, b \geq 1$ and suppose that T_r and W_b are non-empty. Also suppose that

$$W_b \cap T_r = \{v_1,\ldots,v_m\}$$

is non-empty. The set $W_b \cap T_r = W_b \cap T_r\ (p,s,r,b)$ depends only on the four positive integers p, s, r and b.

<u>Lemma 2.15</u> : Suppose that $W_b \cap T_r = \{v_1,\ldots,v_m\}$ is non-empty for some $r, b \geq 1$. Then we have

(i) $v_1 \geq k_1$

(ii) $v_m \leq k_u$

(iii) $(k_1-r) + v_1 \geq k_u+b$

<u>Proof</u> : (i) $v_1 \in W_b \cap T_r$ implies that $v_1 \in T_r$ and by Lemma 2.7

$$v_1 \geq t_1 \geq k_1$$

(ii) $v_m \in W_b \cap T_r$ implies that $v_m \in W_b$ and by Lemma 2.10

$$v_m \leq w_e \leq k_u$$

(iii) Since $v_1 \in T_r$ we have

$$(*) \qquad (s-1)\ (k_1-r) + v_1 \geq p\ .$$

Now assume that $(k_1-r) + v_1 \leq k_u+b$. Then

$$(s-1)\ (k_1-r) + (s-1)\ v_1 \leq (s-1)\ (k_u+b)$$

Hence

$$\begin{aligned}(s-1)\ (k_1-r) + s\ v_1 &\leq (s-2)\ (k_u+b) + v_1 + (k_u+b)\\ &< p + (k_u+b)\end{aligned}$$

Therefore

$$(s-1)\ (k_1-r) + v_1 < p + (k_u+b) - (s-1)\ v_1$$

All we need to show is that $(k_u+b) \leq (s-1)$. Suppose that (k_u+b) is greater than $(s-1)\ v_1$. Then

$$\begin{aligned}(s-2)\ (k_u+b) + v_1 &> (s-2)\ (s-1)\ v_1 + v_1\\ &\geq s\ v_1\\ &> p\end{aligned}$$

which is a contadiction. Thus $(s-1)\ (k_1-r) + v_1 < p$ and this contradicts $(*)$.

Therefore $(k_1-r) + v_1 \geq k_u+b$.

<u>Lemma 2.16</u> : Let s, K, $W_b \cap T_r$ be as defined before. Then for any $c < s-1$ we have

$$c\ (k_1-r) + v_1 > (c-1)\ (k_u+b) + v_m$$

<u>Proof</u> : Assume for a given c, $0 < c < s-1$, that

$$(c-1)\ (k_u+b) + v_m \geq c\ (k_1-r) + v_1$$

Then

$$
\begin{aligned}
(c-1)\ (k_u+b) + v_m + ((s-2)-(c-1))(k_u+b) \\
&\geq c\ (k_1-r) + v_1 + ((s-2)-(c-1))(k_u+b) \\
&\geq c\ (k_1-r) + v_1 + ((s-2)-(c-1))\ (k_1-r) \\
&= (s-1)\ (k_1-r) + v_1 \\
&\geq p
\end{aligned}
$$

by (1). But the left hand side of the above inequality is equal to $(s-2)\ (k_u+b) + v_m$ which is less than p by (4). This is a contradiction.

Therefore $c\ (k_1-r) + v_1 > (c-1)\ (k_u+b) + v_m$.

Lemma 2.17 : Let $L = F / F_{(p)}$ and s, K, $W_b \cap T_r$ be as defined before. Suppose, for $b, r \geq 1$, $W_b \cap T_r$ is non-empty. Let $y \in H_{k_1-r}$, $z \in H_{k_u+b}$ and

$$D = \{y\} \cup (H_{v_1} \cup \ldots \cup H_{v_m}) \cup \{z\} .$$

Then, the set D freely generates a free nilpotent subalgebra of class s in L.

Proof : $D \subseteq H$ and hence it is linearly independent. Suppose D is given the same order as H. Let $\underline{H}$ be a Hall set constructed on on D. Due to Lemma 2.16, a suitable D-length preserving order for $\underline{H}_1 \cup \ldots \cup \underline{H}_{s-1}$ also preserves X-length. Then $\underline{H}_1 \cup \ldots \cup \underline{H}_{s-1}$ becomes a subset of H. In $\underline{H}_{s-1}$, an element f with a maximal X-length satisfies

$$
\begin{aligned}
X\text{-length}\ (f) &= (s-2)\ (k_u+b) + v_m \\
&< p.
\end{aligned}
$$

Hence, $\underline{H}_1 \cup \ldots \cup \underline{H}_{s-1} \subseteq H_1 \cup \ldots \cup H_{p-1}$. Let N be the subalgebra of F generated by D. All we need to show is that $N_{(a)} = \{0\}$ for $a \geqslant s$. In $N_{(s)}$ an element g whose leading term has a minimal X-length is of the form

$$g = \alpha \underbrace{y(y(\ldots(y}_{(s-1)\text{-times}}h))\ldots) + \ldots$$

where $\alpha \in \underline{k}$, $h \in H_{v_1}$. Then

$$\begin{aligned} \text{X-length } (g) &= (s-1)\,(k_1 - r) + v_1 \\ &\geqslant p. \end{aligned}$$

Thus, $N_{(a)} = \{0\}$ for $a \geqslant s$ and D freely generates a free nilpotent subalgebra of class s in L.

<u>Theorem 2.7</u> : Let $L = F / F_{(p)}$ and s, $K = \{k_1, \ldots, k_u\}$ be as above. Suppose that there are integers $b, r \geqslant 1$ for which the set $W_b \cap T_r = \{v_1, \ldots, v_m\}$ is non-empty. Let S be a subalgebra of L with a generating set $Y = \{y, z, y_1, \ldots, y_i, \ldots\}$, such that $y \in L_{(k_1 - r)}$, $y \notin L_{(k_1 - r + 1)}$, $z \in L_{(k_u + b)}$, $z \notin L_{(k_u + b + 1)}$ and $Y - \{y, z\} \subseteq L_{(v_1)}$ and it is linearly independent modulo $L_{(v_m + 1)}$. Then S is free nilpotent of class s in L and Y is a set of free generators for S.

<u>Proof</u> : The case $Y = \{y, z\}$ is covered in Theorems 2.6 and 2.7. Suppose that $|Y| > 2$. We can take without loss of generality, $y \in H_{k_1 - r}$, $z \in H_{k_u + b}$ and S to be finitely generated on generators $Y = \{y, z, y_1, \ldots, y_n\}$ such that

$$y_i = \sum \alpha_j d_j + f_i$$

for $i = 1, \ldots, n$ and $d_j \in D - \{y\}$, $f_i \in L_{(v_m + 1)}$. (D and N are as in the proof of the previous lemma) . Put

$$\sum \alpha_j d_j = a_i$$

for $i = 1,\dots,n$ so that

(*) $$y_i = a_i + f_i$$

Then the set $\{y,z,y_1,\dots,y_n\}$ is a subset of N and it is linearly independent modulo $N_{(2)}$. Thus it generates a free nilpotent subalgebra of class s. Let A be the subalgebra of F geberated by $\{y,z,y_1,\dots,y_n\}$. All we need to show is that $S \cong A / A_{(s)}$.

Let $\sigma : A \to S$ be the map defined by

$$\begin{aligned} \sigma :\ & y \mapsto y \\ & z \mapsto z \\ & a_i \mapsto y_i \end{aligned}$$

for $i = 1,\dots,n$. Extend σ to A. Then

$$A_{(s)} \subseteq \text{Kernel } (\sigma) .$$

Let $\underline{H}_1 \cup \dots \cup \underline{H}_{s-1}$ be a Hall basis for $A / A_{(s)}$. Assume that there exists an element $f \in A_{(w)}$, $f \notin A_{(w+1)}$, $w < s$, such that $\sigma(f) = 0$. Then

$$f = \sum \gamma_j h_j(y,z,a_1,\dots,a_n) + g$$

where $\gamma_j \in \underline{k}$, $h_j \in \underline{H}_w$, $g \in A_{(w+1)}$ and not all γ_j is equal to zero. Since $w < s$, $h_j \neq 0$ in L. We have

$$\sigma(f) = \sum \gamma_j h_j(y,z,y_1,\dots,y_n) + \sigma(g)$$

(†) $$= \sum \gamma_j h_j(y,z,a_1,\dots,a_n) + a + \sigma(g)$$

where a contains terms each of which has at least one element of the form as in $(*)$, and $\sigma(g) \in L_{(w(k_1-r)+v_1)}$. Then, by Lemma 2.16, we have

$$\text{X-length}\ (\ \mathrm{ld}\ (\ \sigma\ (g)\ )\ ) \ > \ \text{X-length}\ (h_j)$$

for all h_j used in the sum above.

Let $\{h_1,\ldots,h_r\}$ be those h_j occuring in $(\dagger)$ whose leading terms have a minimal X-length. Then every other term in $(\dagger)$ has leading terms with greater X-length. Thus $\sigma(f) = 0$ implies that

$$\sum_{j=1}^{r} y_j h_j(y,z,a_1,\ldots,a_n) = 0$$

This contradicts the linear independence of the basis elements of $A / A_{(s)}$. Hence $\sigma(f) \neq 0$ and Kernel $(\sigma) = A_{(s)}$.

Therefore $S \cong A / A_{(s)}$.

Note that the theorems stated above are not mutually exclusive. A set Y may satisfy the hypothesis of more than one of the theorems mentioned . If S is a subalgebra of L such that it is free nilpotent of class s, S need not have a generating set Y satisfying one of the theorems above unless $|Y| = 2$. The following example will illustrate this.

<u>Example 2.9</u> : Let $L = F / F_{(19)}$, $s = 3$. Then $K = \{7,8,9\}$. Suppose that S is the subalgebra of L generated by the set $Y = \{y_1,y_2,y_3\}$, where

$$y_1 = a + b$$

$$y_2 = b$$
$$y_3 = a + c$$

where $a \in H_7$, $b \in H_8$, $c \in H_{10}$. Then, S is free nilpotent of class 3. Y is not linearly independent modulo $L_{(10)}$, $(10 = k_u + 1)$ since

$$y_3 = y_1 - y_2 \quad (\text{modulo } L_{(10)})$$

In fact, Y does not satisfy the hypothesis of any of the theorems mentioned before in this chapter.

Theorem 2.10 : Let $L = F / F_{(p)}$ be a free nilpotent Lie algebra of class p and S be a subalgebra on two generators. Then S is free nilpotent of class s, $s \leq p$, if and only if it has a generating set which satisfies the hypothesis of one or more of the Theorems 2.3, 2.4, 2.5, 2.6 or 2.7.

Proof : If S has a generating set which satisfies the hypothesis of the above mentioned theorems, then S is free nilpotent of class s. Here we prove the other implication.

Let S be a free nilpotent subalgebra of class s in L. Let $K = \{k_1, \ldots, k_u\}$ and for given positive integers r, $b \geq 1$ let

$$T_r = \{t_1, \ldots, t_n\}$$
$$W_b = \{w_1, \ldots, w_e\}$$
$$W_b \cap T_r = \{v_1, \ldots, v_m\}$$

Suppose $Y = \{y_1, y_2\}$ is a free generating set for S. Assume that Y does not satisfy the hypothesis of any of the theorems mentioned above. Then we have one of the following cases occuring.

Case I : Suppose $y_1 \in L_{(k_1-r)}$, $y_1 \notin L_{(k_1-r+1)}$ for some $r \geqslant 1$ and

(i) $y_2 \notin L_{(t_1)}$

(ii) $y_2 \in L_{(t_n+1)}$

Suppose (i) is the case. Take the element g defined by

$$g = \underbrace{y_1(y_1(\ldots(y_1}_{(s-1)\text{-times}}y_2))\ldots)$$

of $S_{(s)} = \{0\}$. If X-length (ld (y_2)) $= q < t_1$, then

$$\begin{aligned} \text{X-length (ld } (g) \text{)} &= (s-1)\ (k_1-r) + q \\ &\geqslant p \end{aligned}$$

since $g = 0$ in L. Also

$$\begin{aligned} (s-2)\ q + (k_1-r) &< (s-2)\ t_1 + (k_1-r) \\ &< p. \end{aligned}$$

Then, by definition of T_r, $q \in T_r$. This contradicts t_1 being the smallest positive integer belonging to T_r. Hence (i) cannot happen.

Suppose (ii) is the case. Let X-length (ld (y_2)) $= q > t_n$. If g is the element of $S_{(s-1)}$ defined by

$$g = \underbrace{y_2(y_2(\ldots(y_1}_{(s-2)\text{-times}}y_2))\ldots)$$

then

$$\begin{aligned} \text{X-length (ld } (g) \text{)} &= (s-2)\ q + (k_1-r) \\ &< p \end{aligned}$$

But since $q > t_n$, we have

$$\begin{aligned} (s-1)\ (k_1-r) + q &> (s-1)\ (k_1-r) + t_n \\ &\geqslant p \end{aligned}$$

Then $q \in T_r$ and this contradicts t_n being the largest element of T_r.

Therefore, Case I cannot happen.

Case II : Suppose $y_2 \in L_{(k_u+b)}$, $y_2 \notin L_{(k_u+b+1)}$ for some $b \geq 1$. Then we have the following alternatives :

(i) $y_1 \notin L_{(w_1)}$

(ii) $y_1 \in L_{(w_e+1)}$

Suppose (i) is the case. Then the element g of $S_{(s)}$ defined by

$$g = \underbrace{y_1(y_1(\dots(y_1}_{(s-1)\text{-times}}y_2))\dots)$$

is zero in L. Thus if X-length (ld (y_1)) $= q < w_1$, then

$$\text{X-length (ld } (g) \text{)} = (s-1)\, q + (k_u+b)$$
$$\geq p\ .$$

But since $q < w_1$, we also have

$$(s-2)\,(k_u+b) + q < (s-2)\,(k_u+b) + w_1$$
$$< p\ .$$

Then $q \in W_b$ and this contradicts w_1 being the smallest element of W_b. Hence, (i) cannot happen.

Now suppose (ii) is the case. Let X-length (ld (y_1)) $= q$ and $q > w_e$. Then, the element g of $S_{(s-1)}$ defined by

$$g = \underbrace{y_1(y_1(\dots(y_1}_{(s-2)\text{-times}}y_2))\dots)$$

is non-zero in L so that

$$\text{X-length (ld } (g) \text{)} = (s-2)\, q + (k_u+b)$$
$$< p$$

But since $q > w_e$ we have

$$(s-1)\,q + (k_u+b) > (s-1)\,w + (k_u+b)$$
$$\geqslant p\ .$$

Then $q \in W_b$ and this contradicts w_e being the largest element of W_b. Hence Case II cannot happen

Therefore if S is a free nilpotent subalgebra of class s on two-generators $Y = \{y_1, y_2\}$, then Y must satisfy the hypothesis of one or more of the theorems mentioned above.

The subalgebras that we have looked at were all free nilpotent of class s, where

$$s-1 = [p/k]$$

for some positive integer k. This need not be the case as the last part of the following example will show, that is, there are subalgebras in a free nilpotent Lie algebra which are themselves free nilpotent of class s and there is no integer k such that $s-1 = [p/k]$.

<u>Conjecture</u> : Let $L = F / F_{(p)}$ and S be a free nilpotent subalgebra of class s in L. If S is not a two-generator subalgebra, then

$$s-1 = [p/k]$$

for some positive integer k.

<u>Example 2.10</u> : In this example we determine all free nilpotent subalgebras of $L = F / F_{(9)}$, where F is a free Lie algebra on free generators X, $|X| \geqslant 2$.

I. Free nilpotent subalgebras of class $s = 2$:

We have $[9/5] = [9/6] = [9/7] = [9/8] = s-1 = 1$. Thus $K = \{5,6,7,8\}$.

(i) If S has generators $Y \subseteq L_{(5)}$ and linearly independent modulo $L_{(9)}$, then S is free nilpotent of class 2.

For given integers $r \geq 1$, the following sets T_r are nonempty :

$$T_1 = \{5,6,7,8\}$$
$$T_2 = \{6,7,8\}$$
$$T_3 = \{7,8\}$$
$$T_4 = \{8\}$$

(ii) If S has generators Y such that $y \in Y$ is the only element in Y satisfying $y \in L_{(5-r)}$, $y \notin L_{(5-r+1)}$ for $r = 1,2,3,4$, and $Y - \{y\} \subseteq L_{(t_1)}$, where t_1 is the smallest element in T_r, and linearly independent in L, then S is free nilpotent of class 2.

If S is a two-generator subalgebra on $Y = \{y_1, y_2\}$ such that $y_1 \in L_{(5-r)}$, $y_1 \notin L_{(5-r+1)}$ and $y_2 \in L_{(5)}$, then S is free nilpotent of class 2.

If S is a free nilpotent subalgebra of class 2 in L, then it satisfies (i) or (ii) above. Hence in this case the classification is complete.

II. Free nilpotent subalgebras of class $s = 3$:

We have $[9/3] = [9/4] = s-1 = 2$. Thus $K = \{3,4\}$.

(i) If S has generators $Y \subseteq L_{(3)}$ and linearly independent modulo $L_{(5)}$, then S is free nilpotent of class 3.

For given $r \geq 1$, the following sets are non-empty :

$$T_1 = \{5,6\}$$
$$T_2 = \{7\}$$

In this case $T_r \cap K = \phi$ and there are only two-generator subalgebras.

(ii) If S is a subalgebra on $Y = \{y_1, y_2\}$ such that $y_1 \in L_{(2)}$, $y_1 \notin L_{(3)}$ and $y_2 \in L_{(5)}$, $y_2 \notin L_{(7)}$, then S is free nilpotent of class 3.

If $y_1 \notin L_{(2)}$ and $y_2 \in L_{(7)}$, $y_2 \notin L_{(8)}$, then again S is free nilpotent of class 3.

For given $b \geq 1$, the following sets are non-empty :

$$W_1 = \{2,3\}$$
$$W_2 = \{2\}$$
$$W_3 = \{1\}$$

In this case $W_b \cap K = \phi$ for $b = 2,3$ but it is non-empty for $b = 1$.

(iii) If S has generators Y such that only $y \in Y$ belongs to $L_{(5)}$, $y \notin L_{(6)}$ and $Y - \{y\} \subseteq L_{(3)}$ and linearly independent modulo $L_{(4)}$, then S is free nilpotent of class 3.

If S is a two-generator subalgebra on $Y = \{y_1, y_2\}$ such that $y_1 \in L_{(w_1)}$, $y_1 \notin L_{(w_e+1)}$ and $y_2 \in L_{(4+b)}$, $y_2 \notin L_{(4+b+1)}$, for $b = 1,2,3$, then S is free nilpotent of class 3.

If S is a subalgebra which is free nilpotent of class 3, it need not be one of the above mentioned algebras. There is one exceptional case :

Let S be a subalgebra on generators $Y = \{y_1, y_2, y_3\}$ such that $Y \subseteq L_{(3)}$ and it is linearly independent modulo $L_{(6)}$

$= L_{(k_u+2)}$ but linearly dependent modulo $L_{(5)}$ as follows :

$$y_1 = a + a'$$
$$y_2 = a'$$
$$y_3 = a + b'$$

where $a, a' \in L_{(3)}$, $a, a' \notin L_{(4)}$ and $b' \in L_{(5)}$, $b' \notin L_{(6)}$. Then

$$y_3 = y_1 - y_2 \pmod{L_{(5)}}$$

but S is free nilpotent of class 3.

III. Free nilpotent subalgebras of class $s = 5$.

We have $[9/2] = s-1 = 4$. Thus $K = \{2\}$.

(i) If S is a subalgebra with generators $Y \subseteq L_{(2)}$ and linearly independent modulo $L_{(3)}$, then S is free nilpotent of class 5.

If on the other hand S is a free nilpotent subalgebra of class 5, then it satisfies (i). Thus the classification is complete for $s = 5$.

IV. Free nilpotent subalgebras of class $s = 9$.

By Theorem 2.3 S is free nilpotent subalgebra of class 9 in L if and only if it has a generating set Y which is linearly independent modulo $L_{(2)}$.

There are no free nilpotent subalgebras in L of class s for $s = 6,7,8$.

All the subalgebras we have looked at up to this point were of class s, where

$$s - 1 = [9/v]$$

for some $v < 9$. There is, however, one exceptional type of subalgebra in L.

There is no integer v such that

$$[9/v] = 3$$

but the following is a two-generator subalgebra which is free nilpotent of class 4 :

Let S be the subalgebra on generators $Y = \{y_1, y_2\}$ such that

$$\text{X-length}(\text{ld}(y_1)) = 2$$

$$\text{X-length}(\text{ld}(y_2)) = 3$$

Then S is a free nilpotent of class 4.

Theorem 2.11 : Let $L = F / F_{(p)}$. Then L has no free nilpotent subalgebra of class s, where

$$p > s > [p/2] + 1 \geqslant 2$$

Proof : Suppose that there is a subalgebra S such that S is free nilpotent of class s with

$$s - 1 > [p/2]$$

Then $2(s-1) \geqslant p$. Hence there can only be one element y in Y such that $y \in L_{(2)}$. Then $Y - \{y\}$ have elements which are non-zero modulo $L_{(2)}$. Suppose this is the case. But in $S_{(s-1)}$, there is the element f defined by

$$f = \underbrace{y(y(\ldots(yy'))\ldots)}_{(s-2)\text{-times}}$$

where $y' \in Y-\{y\}$ and

$$X\text{-length}\ (\ \mathrm{ld}\ (f)\ ) \geq (s-2)\,2 + 1 < p\ .$$

Then we have

$$(*) \qquad p \leq 2\,(s-1) < p+1$$

which implies that $2\,(s-1) = p$. But the element g of $S_{(s)} = \{0\}$ defined by

$$g = \underbrace{y'(y'(\ldots(yy'))\ldots)}_{(s-1)\text{-times}} \quad ,$$

where $y' \in Y-\{y\}$ must have leading term whose X-length is greater than p. Then

$$(s-1)\,1 + 2 \geq p$$

$$(s-1) \geq p - 2$$

$$s \geq p - 1\ .$$

This contradicts (*). Thus there is no y in the generating set Y of S such that $y \in L_{(2)}$ and every element of Y is non-zero modulo $L_{(2)}$.

If for any $y_1, y_2 \in Y$, $\{y_1, y_2\}$ is linearly independent modulo $L_{(2)}$, then $S_{(p-1)}$ would have a non-zero element and this would contradict $s < p$. Thus given any $y_1, y_2 \in Y$ we have

$$y_1 = a + f_1$$

$$y_2 = a + f_2 \quad ,$$

where a is an element of degree 1 in L, i.e. ,

$$a = \sum_j \alpha_j x_j$$

where $x_j \in X$, $f_1, f_2 \in L_{(2)}$. Suppose that

$$X\text{-length} \, (\, ld \, (af_2) \,) = v < X\text{-length} \, (\, ld \, (f_1 a) \,) \quad .$$

Then, we have

$$y_1 y_2 \in L_{(v)}$$

$$y_1 y_2 \notin L_{(v+1)}$$

Let f be the element of $S_{(p-v+2)}$ defined by

$$f = \underbrace{y_1(y_1(\ldots(y_1}_{(p-v+1)\text{-times}} y_2))\ldots) \quad .$$

Then

$$f = \underbrace{a(a(\ldots(a}_{(p-v+1)\text{-times}} f_2))\ldots) \quad (\text{ modulo } L_{(p)}) \quad .$$

and $f \in L_{(p-1)}$. But the element g of $S_{(p-v+2)}$ defined by

$$g = \underbrace{y_2(y_2(\ldots(y_1}_{(p-v+1)\text{-times}} y_2))\ldots)$$

also satisfies

$$g = \underbrace{a(a(\ldots(a}_{(p-v+1)\text{-times}} f_2))\ldots) \quad (\text{ modulo } L_{(p)}) \quad .$$

Therefore in $S \,/\, S_{(s)}$, we have two non-zero elements f and g, $f \neq g$., such that

$$f - g = 0$$

This contradicts S being free nilpotent of class s. Hence the result.

<u>Note</u> : In Lemma 2.7 we stated that $\underline{H}_1 \cup \ldots \cup \underline{H}_{s-1} \subsetneq H$, where $\underline{H}$ is a Hall set constructed on $D = H_{k_1} \cup \ldots \cup H_{k_u}$. This follows from the fact that $2k_1 > k_u$.

Let $h, h' \in D$, $h < h'$, and suppose that $h \in H_{k_i}$, $h' \in H_{k_j}$, $k_1 \leqslant i \leqslant j \leqslant k_u$. Then we can write

$$hh' = h(v_1 v_2)$$

where $v_1, v_2 \in H$, $v_1 < v_2$. Suppose that $h < v_1$. Then we have X-length $(h) \leqslant$ X-length (v_1) and thus

$$k_u \geqslant \text{X-length } (h') \geqslant 2 \text{ X-length } (v_1) \geqslant 2 \text{ X-length } (h) \geqslant 2k_1$$

This contradicts $k_u < 2k_1$. Hence $h \geqslant v_1$ and $hh' \in H$.

Thus $\underline{H}_2 \subseteq H$. One can similarly show that

$$\underline{H}_1 \cup \ldots \cup \underline{H}_{s-1} \subsetneq H$$

Chapter 3

ON SUBALGEBRAS OF FREE POLYNILPOTENT LIE ALGEBRAS

A subalgebra of a free polynilpotent Lie algebra is itself polynilpotent possibly relative to a different sequence than the original algebra. In the first section of this chapter we prove some general theorems on free polynilpotent Lie algebras. In the second section we look at two-generator subalgebras, which will be followed by a study of more general subalgebras. In section four we determine those subalgebras which are themselves free polynilpotent but not necessarily relative to the same sequence as the original Lie algebra.

§ 1. Some General Theorems on Free Polynilpotent Lie Algebras

Let F be a free Lie algebra over a field $\underline{k}$ with a free generating set X, where $|X| \geqslant 2$. In this chapter

$$L = F / F_{(n_1),\ldots,(n_k)}$$

will denote a free polynilpotent Lie algebra on free generators X relative to the sequence $\{n_1,\ldots,n_k\}$, where $n_i \geqslant 2$ for $i = 1,\ldots,k$. If $k = 1$, then we have a free nilpotent Lie algebra of class n_1.

Theorem 3.1 : Let $L = F / F_{(n_1),\ldots,(n_k)}$ and $k \geqslant 2$. Then for any positive integer m

$$L_{(m)} \neq \{0\}$$

Proof : Let X be a well-ordered free generating set for L, $|X| \geq 2$, and $x_1, x_2 \in X$, where $x_1 < x_2$. Let A be the set defined by

$$A = \left\{ g = \underbrace{x_1(x_1(\ldots(x_1}_{t\text{-times}} x_2))\ldots) \;:\; t \geq 1 \right\}$$

Obviously $A \cap L_{(m)}$ is non-empty for any positive integer $m \geq 2$. But $A \subseteq C_2$, the free generating set for $F_{(2)}$ and thus

$$A \cap L_{(n_1),(n_2)} = \phi$$

which implies that the elements of A are all non-zero in L.

Therefore $L_{(m)} \neq \{0\}$ for any $m \geq 1$.

Corollary 3.1 : Let $L = F / F_{(n_1),(n_2),\ldots,(n_k)}$, where $k \geq 2$. Then

$$L_{(n_1),\ldots,(n_q),(m)} \neq \{0\}$$

for any $m \geq 1$ and $q < n_{k-1}$.

Proof : Let $G = F_{(n_1),\ldots,(n_q)}$. Then

$$G / G_{(n_{q+1}),\ldots,(n_k)}$$

is a free polynilpotent Lie algebra and $q < k-1$ implies that $G_{(m)} \neq \{0\}$ for any $m \geq 1$.

We have shown in Chapter 1, § 8 that the set B defined as

$$B = B_1 \cup B_2 \cup \ldots \cup B_k \quad ,$$

where

$$B_i = H_1^{Cn_1,\ldots,n_{i-1}} \cup \ldots \cup H_{n_{i-1}}^{Cn_1,\ldots,n_{i-1}}$$

for $i = 1,\ldots,k$, forms a basis for $L = F / F_{(n_1),\ldots,(n_k)}$.
The following result is stated in [20] :

Lemma 3.1 : Let L be a free polynilpotent Lie algebra relative to the sequence $\{n_1,\ldots,n_k\}$. Then the factors $L_{(m)} / L_{(m+1)}$ of the lower central series of L are free abelian Lie algebras. The elements in $B_1 \cup B_2 \cup \ldots \cup B_k$ of X-length = m forms an additive basis for $L_{(m)} / L_{(m+1)}$.

Corollary 3.2 : The factors

$$L_{(n_1),\ldots,(n_q),(m)} / L_{(n_1),\ldots,(n_q),(m+1)}$$

where $q < k$, are free abelian and those elements in $B_1 \cup \ldots \cup B_k$ of $C_{n_1,\ldots,n_q}$-length = m forms an additive basis for it.

As a corollary to Theorem 3.1 and Corollary 3.1 we have the important result :

Theorem 3.2 : Let S be a subalgebra of the free polynilpotent Lie algebra L. Then S is nilpotent if and only if $S \subseteq L_{(n_1),\ldots,(n_{k-1})}$.

To decide when a nilpotent subalgebra is free nilpotent one can use the methods of the previous chapter.

§ 2. Two-Generator Subalgebras of a Free Polynilpotent Lie Algebra

In this section we will use and generalize the methods and results of Chapter 3, § 1.

Theorem 3.3 : Let $L = F / F_{(n_1),\ldots,(n_k)}$ and S be a two-generator subalgebra of L on generators $Y = \{y_1, y_2\}$. Suppose that

$$y_1 \in L_{(m_1),\dots,(n_q),(m)}\ , \quad y_1 \notin L_{(n_1),\dots,(n_q),(m+1)}$$

$$y_2 \in L_{(n_1),\dots,(n_q),(e)}, \quad y_2 \notin L_{(n_1),\dots,(n_q),(e+1)}$$

where $q < k$, $m \leqslant e < n_{q+1}$ and $\{y_1, y_2\}$ is linearly independent modulo $L_{(n_1),\dots,(n_q),(p+1)}$, $e \leqslant p \leqslant n_{q+1}$, and p is the smallest such integer. Then S is a polynilpotent subalgebra of L relative to the sequence $\{s, n_{q+2}, \dots, n_k\}$, where s is the smallest integer satisfying

$$(s-1)\, m \ + \ p \ \geqslant \ n_{q+1}$$

<u>Proof</u> : If $m = e$, then $p = e$ since in that case Y would be linearly independent modulo $L_{(n_1),\dots,(n_q),(e+1)}$. If $m = e$, then $p \geqslant e$.

In $S_{(s)}$, an element whose leading term has a minimal $C_{n_1,\dots,n_q}$-length is of the form

$$g \ = \ \alpha\, \underbrace{y_1(y_1(\dots(y_1}_{(s-1)\text{-times}} y_2))\dots) \ + \ \dots$$

where $\alpha \in \underline{k}$. Since

$$C_{m_1,\dots,n_q}\text{-length } (\ \mathrm{ld}\ (g)\ ) \ = \ (s-1)\ m \ + \ p$$
$$\geqslant \ n_{q+1}$$

it follows that $g \in L_{(n_1),\dots,(n_{q+1})}$. Thus

$$S_{(s)} \subseteq L_{(n_1),\dots,(n_{q+1})} \ .$$

This implies that

$$(*) \qquad S_{(s),(n_{q+2}),\dots,(n_k)} \subseteq L_{(n_1),\dots,(n_k)} = \{0\}$$

Further more, we have

$$\mathrm{ld}\,(g) \in C_{n_1,\dots,n_{q+1}}$$

so that $g \notin L_{(n_1),\dots,(n_{q+2}),(2)}$. If $g' \in S_{(s)}$ is defined by

$$g' = \beta \underbrace{y_2(y_1(\dots(y_1}_{(s-1)\text{-times}} y_2))\dots) + \dots$$

where $\beta \in \underline{k}$, then g' also satisfies

$$g' \in L_{(n_1),\dots,(n_{q+1})} , \quad g' \notin L_{(n_1),\dots,(n_{q+1}),(2)}$$

Hence if any of the integers $n_{q+2},\dots,n_k$ is replaced by a smaller integer, then there would be a non-zero element in $S_{(s),(n_{q+2}),\dots,(n_k)}$ and an inclusion of the form $(*)$ would not be true.

Similarly, in $S_{(s-1)}$ the element

$$f = \underbrace{y_1(y_1(\dots(y_1}_{(s-2)\text{-times}} y_2))\dots) + \dots$$

satisfies

$$C_{n_1,\dots,n_q}\text{-length}\,(\,\mathrm{ld}\,(f)\,) = (s-2)\,m + p$$
$$< n_{q+2}$$

Hence $f \notin L_{(n_1),\dots,(n_{q+1})}$. Then $fg \notin L_{(n_1),\dots,(n_{q+2}),(2)}$ which implies that

$$S_{(s-1),(2)} \nsubseteq L_{(n_1),\dots,(n_{q+2}),(2)}$$

Proceeding similarly one concludes that $S_{(s-1),(n_{q+2}),\ldots,(n_k)}$ has a non-zero element in L. Hence s cannot be replaced by a smaller integer and an inclusion of the type (*) still be true.

Therefore S is polynilpotent relative to the sequence $\{s, n_{q+2}, \ldots, n_k\}$.

We now generalize this to :

<u>Theorem 3.4</u> : Let $L = F / F_{(n_1),\ldots,(n_k)}$ and S be a two-generator subalgebra of L on generators $Y = \{y_1, y_2\}$. Suppose that

$$y_1 \in L_{(n_1),\ldots,(n_p),(m)}, \quad y_1 \notin L_{(n_1),\ldots,(n_p),(m+1)}$$

$$y_2 \in L_{(n_1),\ldots,(n_q),(e)}, \quad y_2 \notin L_{(n_1),\ldots,(n_q),(e+1)}$$

where $p < q < k$, $m < n_{p+1}$, $e < n_{q+1}$. Then S is [redacted] poly-nilpotent relative to the sequence

$$\{2, [n_{q+1}/e] + 1, n_{q+2}, \ldots, n_k\}$$

<u>Proof</u> : Let

$$ld\,(y_1) = h_1 \in H_m^{Cn_1,\ldots,n_p}$$

$$ld\,(y_2) = h_2 \in H_e^{Cn_1,\ldots,n_q}$$

Then

$$ld\,(y_1 y_2) \in L_{(n_1),\ldots,(n_q),(e)}$$

but

$$ld\,(y_1 y_2) \notin L_{(n_1),\ldots,(n_q),(e+1)}$$

since $C_{n_1,\dots,n_q}$-length $(h_1) = 0$, i.e. , $h_1 \notin L_{(n_1),\dots,(n_q)}$, which implies that $h_1 h_2 \notin L_{(n_1),\dots,(n_q),(e+1)}$. Hence

$$S \not\subseteq L_{(n_1),\dots,(n_q),(e)}$$

$$S_{(2)} \subseteq L_{(n_1),\dots,(n_q),(e)}$$

Then

$$S_{(2),([n_{q+1}/e])} \not\subseteq L_{(n_1),\dots,(n_{q+1})}$$

but

$$S_{(2),([n_{q+1}/e]+1)} \subseteq L_{(n_1),\dots,(n_{q+1})} .$$

Furthermore ,

$$S_{(2),([n_{q+1}/e]+1)} \not\subseteq L_{(n_1),\dots,(n_{q+1}),(2)}$$

Thus

$$S_{(2),([n_{q+1}/e]+1),(n_{q+2}),\dots,(n_k)} = \{0\}$$

in L, but if any of the integers $2, [n_{q+1}/e], n_{q+2},\dots,n_k$ is replaced by a smaller integer this is not true.

Therefore S is polynilpotent relative to the sequence $\{2, [n_{q+1}/e]+1, n_{q+2},\dots,n_k\}$.

§ 3. Subalgebras of a Free Polynilpotent Lie Algebra in General

We now generalize the results of Chapter 2, § 2 to cover the polynilpotent case. Let $L = F / F_{(n_1),\dots,(n_k)}$, where F is a free Lie algebra on free generators X. If f is an arbitrary element of L, then $ld\ (f) \in B$, the additive basis of L defined in Chapter 1. The set B is given a total order taking into consideration the following in decreasing order of priority :

$$C_{n_1,\ldots,n_{k-1}}\text{-lengths}$$

$$C_{n_1,\ldots,n_{k-2}}\text{-lengths}$$

$$\vdots$$

$$C_{n_1}\text{-lengths}$$

$$X\text{-lengths}$$

We will use the term "generalized length" when comparing elements of B with respect to this ordering.

<u>Lemma 3.2</u> : Let S be a subalgebra of L and Y be a generating set for S. Let $y \in Y$ be such that $ld\ (y)$ is minimal among the leading terms of the elements of Y. Suppose $y \in H_m^{Cn_1,\ldots,n_q}$ for some $q < k$, $m < n_{q+1}$. Then the integers m and q are independent of the choice of the generating set Y and hence they are invariants of the subalgebra S of L.

<u>Proof</u> : Suppose that S has a generating set Y' with an element $y' \in H_{m'}^{Cn_1,\ldots,n_{q'}}$ having a minimal leading term among the leading terms of the elements of Y' and assume that $q = q'$ and $m = m'$ are not both true. If $q = q'$ and $m < m'$, then there is an element in S (namely $y \in Y$) which cannot be expressed in terms of the generating set Y', since $y \notin L_{(n_1),\ldots,(n_q),(m')}$. Thus we must have $m = m'$.

If $q \neq q'$, then a similar contradiction arises.

Therefore the integers m and q are invariants of the subalgebra S of L.

Lemma 3.3 : Let S be a subalgebra of L and m, q be as defined before. Then one can choose a pair of elements $\{y_1, y_2\}$ in Y such that

(i) $\{y_1, y_2\}$ is linearly independent modulo $L_{(n_1),\ldots,(n_t),(e)}$ where $t \geq q$ is the smallest such integer, $e < n_{t+1}$ and if $t = q$, then $e \geq m$ is the smallest such integer. If $\{y_i, y_j\}$ is any pair of elements of Y distinct from $\{y_1, y_2\}$, and $\{y_i, y_j\}$ is linearly independent modulo $L_{(n_1),\ldots,(n_{t'}),(e'+1)}$, then either $t' > t$ or $t = t'$ and $e' \geq e$.

(ii) The "generalized length" of the leading term of $y_1 y_2$ is less than or equal to that of any other such pair from Y satisfying (i).

Proof : Let Y be a generating set for S with invariants q and m as defined before. This means that there is an element y_1 in Y whose leading term is minimal among the leading terms of elements of Y and

$$\mathrm{ld}\,(y_1) \in H_m^{Cn_1,\ldots,n_q}$$

We prove the lemma by choosing a pair $\{y_1, y_2\}$ which satisfies (i) and (ii).

Suppose that y_1 is the only element in Y whose leading term belongs to $H_m^{Cn_1,\ldots,n_q}$. Then choose y_2 such that $\mathrm{ld}\,(y_2)$ is minimal among the leading terms of the elements in $Y - \{y_1\}$.

Then obviously $\{y_1, y_2\}$ satisfies (i) and (ii).

Now suppose that there are more than one element, say $\{y_1, y_2, \ldots\} = \underline{Y} \subseteq Y$ such that

$$\mathrm{ld}\,(y_i) \in H_m^{Cn_1,\ldots,n_q}$$

for $y_i \in \underline{Y}$. For each $y_i \in \underline{Y}$, there exists an element $y \in Y$ such that $\{y_i, y\}$ is linearly independent modulo $L_{(n_1),\ldots,(n_p),(u+1)}$, where p, u are minimal such integers, minimality of p having the priority. The element y may, of course, belong to $\underline{Y}$. Let $\{y_j, y\}$, $y_j \in \underline{Y}$, be a pair such that it is linearly independent modulo $L_{(n_1),\ldots,(n_t),(e+1)}$, where t, e are smallest such integers among all such pairs, minimality of t having the priority. We now show that $\{y_j, y\}$ satisfies (i) and (ii).

Suppose that there exists a pair $\{a,b\} \subseteq Y$ such that $a,b \notin \underline{Y}$ and $\{a,b\}$ is linearly independent modulo $L_{(n_1),\ldots,(n_{t'}),(e'+1)}$ where $q \leq t' \leq t$ and if $t' = t$, then $e' \leq e$, and $\{a,b\}$ satisfies (i) and (ii). Let us suppose that $\mathrm{ld}\,(a) \leq \mathrm{ld}\,(b)$. Then for any $y_j \in \underline{Y}$, the pair $\{y_j, a\}$ is also linearly independent modulo $L_{(n_1),\ldots,(n_{t'}),(e'+1)}$ and

$$\mathrm{ld}\,(y_j a) < \mathrm{ld}\,(ab)$$

which contradicts our assumptions about $\{a,b\}$. Hence in any pair satisfying (i) and (ii), one of the elements must belong to $\underline{Y}$. This proves the lemma.

Let us call a pair $\{y_1, y_2\} \subseteq Y$ as described above a "minimal pair" in Y. Obviously a minimal pair need not be unique.

Lemma 3.4 : Let S be a subalgebra of L with the invariants q and m as defined before. Suppose that S has a generating set Y with a minimal pair $\{y_1, y_2\}$ which is linearly independent modulo $L_{(n_1),\ldots,(n_t),(e+1)}$, where $q \leqslant t < k$ and if $q = t$, then $m \leqslant e < n_{q+1}$. Then the integers t and e are independent of the choice of the generating set Y and hence they are invariants of S.

Proof : Suppose Y' is another generating set for S with a minimal pair $\{y_1', y_2'\}$ which is linearly independent modulo $L_{(n_1),\ldots,(n_{t'}),(e'+1)}$, where $t = t'$ and $e = e'$ are not both true. If $t' > t$, then the element $y_1 y_2$ of $S_{(2)}$ cannot be expressed in terms of products of elements in Y'. If $t' = t$, and $e' > e$, a similar contradiction arises.

Hence, we have $t' = t$ and $e' = e$.

For any subalgebra S of $L = F / F_{(n_1),\ldots,(n_k)}$ one can associate two pairs of integers

$$S = S\,(\,(q,m),(t,e)\,)$$

where $q \leqslant t < k$ and if $q = t$, then $m \leqslant e$.

Let us recall that in a minimal pair $\{y_1, y_2\} \subseteq Y$, if $\mathrm{ld}\,(y_2) \notin H_m^{Cn_1,\ldots,n_q}$, then $\mathrm{ld}\,(y_2) \in H_e^{Cn_1,\ldots,n_t}$ and in this case $\{y_1, y_2\}$ is automatically linearly independent modulo $L_{(n_1),\ldots,(n_t),(e+1)}$.

Theorem 3.5 : Let $S\,(\,(q,m),(t,e)\,)$ be a subalgebra of $L = F / F_{(n_1),\ldots,(n_k)}$. Then

(i) if $q = t$, S is polynilpotent relative to the sequence $\{s, n_{q+2}, \ldots, n_k\}$, where s is the smallest integer satisfying

$$(s-1)\, m + e \geq n_{q+1}$$

(ii) if $q < t$, then S is polynilpotent relative to the sequence

$$\{2, [n_{q+1}/e] + 1, n_{t+2}, \ldots, n_k\}$$

<u>Proof</u> : Let Y be a generating set for S and $\{y_1, y_2\}$ be a minimal pair in Y such that

$$\mathrm{ld}\,(y_1) \in H_m^{Cn_1, \ldots, n_q}$$

and $\{y_1, y_2\}$ is linearly independent modulo $L_{(n_1), \ldots, (n_t), (e+1)}$

(i) Suppose that $q = t$ and $e \geq m$. Then, by Lemma 3.3, S has a subalgebra generated by $\{y_1, y_2\}$ which is polynilpotent relative to the sequence $\{s, n_{q+2}, \ldots, n_k\}$, where s is the smallest integer satisfying

$$(s-1)\, m + e \geq n_{q+1}$$

Then

$$S_{(s), (n_{q+2}), \ldots, (n_k)} = \{0\} \qquad (*)$$

due to the minimality of the pair $\{y_1, y_2\}$. If any of the positive integers $s, n_{q+2}, \ldots, n_k$ is replaced by a smaller positive integer an equation of the type $(*)$ is not true.

Therefore, S is polynilpotent relative to the sequence $\{s, n_{q+2}, \ldots, n_k\}$.

(ii) Suppose $q < t$. Then the subalgebra of S generated by $\{y_1, y_2\}$ is polynilpotent relative to the sequence

$$\{2, [n_{t+1}/e] + 1, n_{t+2}, \ldots, n_k\}$$

Furthermore the minimality of the pair $\{y_1, y_2\}$ implies that

$$S_{(2),([n_{t+1}/e]+1),(n_{t+2}),\ldots,(n_k)} = \{0\} \tag{†}$$

If any of the integers $2, [n_{t+1}/e] + 1, n_{t+2}, \ldots, n_k$ is replaced by a smaller integer then an equation of the type (†) would not be true for S since it would not be true for the subalgebra of S generated by $\{y_1, y_2\}$.

Therefore S is polynilpotent relative to the sequence

$$\{2, [n_{t+1}/e] + 1, n_{t+2}, \ldots, n_k\}$$

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§ 4. Free Polynilpotent Subalgebras of a Free Polynilpotent Lie Algebra

In general subalgebras of a free polynilpotent Lie algebra need not be free polynilpotent. In this section we will determine those subalgebras of a free polynilpotent Lie algebra which are themselves free polynilpotent possibly relative to a different sequence of integers than the original algebra.

The following result is due to A. I. Smel'kin [20] :

" Let G be a free polynilpotent group relative to the sequence $\{n_1, \ldots, n_k\}$. A non-cyclic subgroup U of it is itself free polynilpotent relative to the same sequence if and only if it has a system of generators that are linearly independent relative to the derived group of G. This system of generators becomes a free generating set for G. "

A more complete result is given in the case of free soluble gruops :

" Let G be a k-step free soluble group, i.e., $G = F / \delta^k F = F / F_{\underbrace{(2),\dots,(2)}_{k\text{-times}}}$. A subgroup U of it is itself a reduced free group if it possesses a system of generators Y which lie in $\delta^i G$ and linearly independent modulo $\delta^{i+1} G$. In this case U is free soluble of class $k-i$ and Y is a set of free generators for U."

We will generalize Smel'kin's results and give a more complete answer for the free polynilpotent case.

Let F be a free Lie algebra over a field $\underline{k}$ and on a free generating set X', where $|X'| \geq 2$. Let

$$L = F / F_{(n_1),\dots,(n_k)}$$

and let us assume that $n_i \geq 2$ for $i = 1,\dots,k$.

<u>Theorem 3.6</u> : Let S be a subalgebra of L. Then S is free polynilpotent relative to the same sequence as L if and only if S possesses a set of generators which is linearly independent modulo $L_{(2)}$.

<u>Proof</u> : A subset T of L generates a free polynilpotent subalgebra relative to $\{n_1,\dots,n_k\}$ if and only if every finite subset of T containing more than one element does the same.

Hence it is sufficient to consider finitely generated subalgebras only.

Suppose S possesses a generating set $Y' = \{ y'_1, y'_2, \ldots, y'_m \}$ which is linearly independent modulo $L_{(2)}$. Let

$$y'_i = \sum_j \alpha_j x'_j + f'_i$$

for $i = 1, \ldots, m$, where $\alpha_j \in \underline{k}$, $x'_j \in X'$ and $f'_i \in L_{(2)}$. Then there exists a free generating set X for F such that Y' can be transformed onto a generating set $Y = \{ y_1, \ldots, y_m \}$ for S whose elements are of the form

$$(*) \qquad y_i = x_i + f_i$$

where $x_i \in X$ and $f_i \in L_{(2)}$ for $i = 1, \ldots, m$. Suppose that

$$\{ x_1, \ldots, x_m \} = \underline{X} \subseteq X .$$

Let A be the free polynilpotent subalgebra of L relative to the sequence $\{ n_1, \ldots, n_k \}$ generated by $\underline{X}$. We now show that

$$S \cong A / A_{(n_1), \ldots, (n_k)} .$$

Let $\sigma : A \to S$ be the map defined by

$$\sigma : x_i \mapsto y_i \quad \text{for } i = 1, \ldots, m$$

Clearly σ can be extended to A. We now prove that Kernel of σ is equal to $A_{(n_1), \ldots, (n_k)}$. Naturally we have

$$\sigma (A_{(n_1), \ldots, (n_k)}) = \{ 0 \}$$

in L and hence

$$A_{(n_1),\dots,(n_k)} \subseteq \text{Kernel } (\sigma) .$$

Now suppose that there is an element $g \in A_{(n_1),\dots,(n_q),(e)}$, $g \notin A_{(n_1),\dots,(n_q),(e+1)}$, $q < k$, $e < n_{q+1}$, such that $\sigma(g) = 0$. We can express g as

$$g = \sum_j \alpha_j h_j(x_1,\dots,x_m) + d$$

where $\alpha_j \in \underline{k}$, $h_j(x_1,\dots,x_m) \in H_e^{Cn_1,\dots,n_q}$ (H is the Hall basis on X) and $d \in A_{(n_1),\dots,(n_q),(e+1)}$. Then

$$\sigma(g) = \sum_j \alpha_j h_j(y_1,\dots,y_m \text{[redacted]}) + \sigma(d)$$

where $\sigma(d) \in S_{(n_1),\dots,(n_q),(e+1)} \subseteq L_{(n_1),\dots,(n_q),(e+1)}$.
From (*) we deduce that

$$h_j(y_1,\dots,y_m, \text{[redacted]}) = h_j(x_1,\dots,x_m) + b_j$$

where $h_j(x_1,\dots,x_m) \in H_e^{Cn_1,\dots,n_q}$ and each summand of b_j contains at least one element of the form $f_i \in L_{(2)}$ as in (*).
We can write

$$b_j = b_j' + b_j'' ,$$

where $b_j' \notin L_{(n_1),\dots,(n_q),(e+1)}$ and $b_j'' \in L_{(n_1),\dots,(n_q),(e+1)}$.
Let

$$b' = \sum_j b_j' \quad \text{and} \quad b'' = \sum_j b_j''$$

Then

$$\sigma(g) = \sum_j \alpha_j h_j(x_1,\ldots,x_m) + b' + b'' + \sigma(d)$$

and $\sigma(g) = 0$ implies that

$$(\dagger) \qquad \sum_j \alpha_j h_j(x_1,\ldots,x_m) + b' = 0$$

Let $\{h_1,\ldots,h_r\}$ be those $h_j(x_1,\ldots,x_m)$ used in $(\dagger)$ which have a minimal X-length. Then every other term in $(\dagger)$ has greater X-length. Thus we have

$$\sum_{i=1}^{r} \alpha_i h_i(x_1,\ldots,x_m) = 0$$

This contradicts the linear independence of the elements of H. Thus $\sigma(g) \neq 0$ and Kernel $(\sigma) = A_{(n_1),\ldots,(n_k)}$.

Therefore $S \cong A / A_{(n_1),\ldots,(n_k)}$.

Conversely assume that S is a free polynilpotent subalgebra of L relative to $\{n_1,\ldots,n_k\}$. Let Y be any free generating set for S. Suppose there exists an element $y \in Y$ such that $y \in L_{(2)}$. Then for any $y' \in Y$, $y' \neq y$, the elements g_1, g_2 defined by

$$g_1 = \underbrace{y'(y'(\ldots(y'}_{(n_1-2)\text{-times}}y))\ldots)$$

$$g_2 = y(\underbrace{y'(y'(\ldots(y'}_{(n_1-1)\text{-times}}y))\ldots)$$

both belong to $L_{(n_1)}$ and the elements

$$\underbrace{g_1(g_1(\dots(g_1}_{(n_2-1)\text{-times}}g_2))\dots)$$

$$\underbrace{g_2(g_2(\dots(g_1}_{(n_2-1)\text{-times}}g_2))\dots)$$

of $S_{(n_1-1),(n_2)}$ both belong to $L_{(n_1),(n_2)}$. Proceeding similarly one constructs an element of $S_{(n_1-1),(n_2),\dots,(n_k)}$ which belongs to $L_{(n_1),\dots,(n_k)} = \{0\}$. This contradicts S being free polynilpotent relative to the sequence $\{n_1,\dots,n_k\}$. Thus there is no element in Y which belongs to $L_{(2)}$.

Now suppose that Y is not linearly independent modulo $L_{(2)}$. Then there exists $y \in Y$ such that

$$y = \sum_{\substack{j \\ y_j \neq y}} \beta_j y_j + L_{(2)}$$

where $\beta_j \in \underline{k}$ and $y_j \in Y$. Put

$$z = \sum_{\substack{j \\ y_j \neq y}} \beta_j y_j$$

Then $yz \in L_{(3)}$ and the elements $f_1, f_2 \in S_{(n_1-1)}$ defined by

$$f_1 = y(y(\dots(yz))\dots)$$
$$f_2 = \underbrace{z(y(\dots(y}_{(n_1-2)\text{-times}}z))\dots)$$

both belong to $L_{(n_1)}$. Then as in the previous part one can construct an element of $S_{(n_1-1),(n_2),\dots,(n_k)}$ which is zero in L. This contradicts S being free polynilpotent relative to the sequence $\{n_1, \dots, n_k\}$.

Therefore Y is linearly independent modulo $L_{(2)}$.

Corollary 3.3 : Let F be a free Lie algebra on X and $L = F / F_{(n_1),\dots,(n_k)}$. If S is a subalgebra of L which is free polynilpotent relative to the same sequence as L, then there is a subset X' of X such that S is isomorphic to the subalgebra of L generated by X'.

Definition 3.1 : For a given positive integer $s < n_{q+1}$, define $K = \{k_1, \dots, k_u\}$ to be the maximal set of consecutive positive integers satisfying

$$[n_{q+1}/k_i] = s - 1$$

for $i = 1, \dots, u$.

The set K is determined by n_{q+1} and s. Properties of such sets were studied in the previous chapter.

Lemma 3.5 : Let $L = F / F_{(n_1),\dots,(n_k)}$ and $K = \{k_1, \dots, k_u\}$ be as defined above for s and n_{q+1}. Then the subset D of B_{q+1} defined by

$$D = (H_{k_1}^{Cn_1,\dots,n_q} \cup \dots \cup H_{k_u}^{Cn_1,\dots,n_q})$$

freely generates a free polynilpotent subalgebra of L relative to the sequence

$$\{ s, n_{q+2}, \dots, n_k \}$$

and one can choose a basis $\underline{B}$ for this subalgebra such that $\underline{B} \subseteq B$.

<u>Proof</u> : $C_{n_1,\ldots,n_q,k_1}$ is a set of free generators for the subalgebra $F_{(n_1),\ldots,(n_q),(k_1)}$ of F and

$$H_{k_1}^{Cn_1,\ldots,n_q} \cup \ldots \cup H_{2k_1-1}^{Cn_1,\ldots,n_q}$$

is a subset of $C_{n_1,\ldots,n_q,k_1}$. Since $2k_1 > k_u$, we have

$$D \subseteq C_{n_1,\ldots,n_q,k_1} \quad .$$

Hence D is a set of free generators in the subalgebra N of F that it generates. All we need to show is that

$$N_{(s),(n_{q+2}),\ldots,(n_k)} = \{0\}$$

in L and if f is a non-zero element of $N / N_{(s),(n_{q+2}),\ldots,(n_k)}$ then $f \neq 0$ in L. By Lemma 2.7, N is a free nilpotent subalgebra of class s in $L / L_{(n_1),\ldots,(n_{q+1})}$. Hence

$$N_{(s)} \subseteq L_{(n_1),\ldots,(n_{q+1})} \quad .$$

Then

$$N_{(s),(n_{q+2}),\ldots,(n_k)} \subseteq L_{(n_1),\ldots,(n_k)} = \{0\} \ .$$

D is a subset of B and suppose the ordering given to D coincides with that of B. Let $\underline{H}$ be a Hall set constructed on D. Then by giving $\underline{H}_1 \cup \ldots \cup \underline{H}_{s-1}$ an order which coincides with that of B we have

$$(*) \qquad \underline{H}_1 \cup \ldots \cup \underline{H}_{s-1} \subseteq (H_{k_1}^{Cn_1,\ldots,n_q} \cup \ldots \cup H_{n_{q+1}-1}^{Cn_1,\ldots,n_q})$$

Let $\underline{C}_s$ be a set of free generators for $N_{(s)}$ defined as follows :

$$\underline{C}_s = \left\{ f = a_1 a_2 \in \underline{H} \ : \ a_1, a_2 \in \underline{H} \ ; \ \text{D-length } (f) \geq s \right.$$
$$\left. \text{D-length } (a_1) < s \right\}$$

Then

$$\underline{C}_s \subseteq C_{n_1, \ldots, n_{q+1}}$$

since by (*) the orders on $\underline{H}_1 \cup \ldots \cup \underline{H}_{s-1}$ and $H^{C n_1, \ldots, n_q}$ coincide. and $s\,k_1 \geq m_{q+1}$. The last inclusion implies that

$$N_{(s)} \cap L_{(n_1), \ldots, (n_{q+1}), (2)} = \{0\}$$

Thus one can proceed to form a basis $\underline{B}$ for the quotient algebra $N / N_{(s), (n_{q+2}), \ldots, (n_k)}$ such that $\underline{B} \subseteq B$.

Therefore, D generates a free polynilpotent subalgebra of L relative to the sequence $\{ s, n_{q+2}, \ldots, n_k \}$.

Theorem 3.7 : Let $L = F / F_{(n_1), \ldots, (n_k)}$ and S be a subalgebra of L. Suppose that S possesses a generating set $Y \subseteq L_{(n_1), \ldots, (n_q), (k_1)}$ and linearly independent modulo $L_{(n_1), \ldots, (n_q), (k_u + 1)}$, where $q < k$, and $K = \{ k_1, \ldots, k_u \}$ is as defined above. Then S is free polynilpotent relative to the sequence $\{ s, n_{q+2}, \ldots, n_k \}$ and Y is a set of free generators for S.

Proof : As mentioned in the proof of Theorem 3.6 we can take S to be finitely generated without loss of generality. Let D and N be as in the proof of the previous lemma, and let $Y = \{ y_1, \ldots, y_m \}$ be a generating set for S. Then

$$y_i = \sum_j \alpha_j d_j + f_i$$

where $\alpha_j \in \underline{k}$, $d_j \in D$ and $f_i \in L_{(n_1),\ldots,(n_q),(k_u+1)}$. Put

$$a_i = \sum_j \alpha_j d_j$$

so that

$$(*) \qquad y_i = a_i + f_i$$

for $i = 1,\ldots,m$. Then $\{a_1,\ldots,a_m\} \subseteq N$ and it is linearly independent modulo $N_{(2)}$. By Theorem 3.6, $\{a_1,\ldots,a_m\}$ freely generates a free polynilpotent Lie algebra relative to the sequence $\{s, n_{q+2},\ldots,n_k\}$ in L. Let A be the subalgebra of F generated by $\{a_1,\ldots,a_m\}$. We now show that

$$S \cong A / A_{(s),(n_{q+2}),\ldots,(n_k)}$$

Let $\sigma : A \to S$ be the map defined by

$$\sigma : a_i \to y_i$$

for $i = 1,\ldots,m$. Obviously

$$\sigma (A_{(s),(n_{q+2}),\ldots,(n_k)}) = \{ 0 \}$$

so that

$$A_{(s),(n_{q+2}),\ldots,(n_k)} \subseteq \text{Kernel } (\sigma)$$

Let B' be a Hall basis for $A / A_{(s),(n_{q+2}),\ldots,(n_k)}$ constructed on $\{a_1,\ldots,a_m\}$. Suppose that there exists a non-zero element $g \in A / A_{(s),(n_{q+2}),\ldots,(n_k)}$ such that $\sigma(g) = 0$. Then

$$g = \sum_j \alpha_j h_j(a_1,\ldots,a_m) \quad (\text{modulo } A_{(s),(n_{q+2}),\ldots,(n_k)})$$

where $\alpha_j \in \underline{k}$, $h_j(a_1,\dots,a_m) \in B'$ and not all α_j are equal to zero. Then

$$\sigma(g) = \sum \alpha_j h_j(y_1,\dots,y_m) \quad (\text{ modulo } F_{(n_1),\dots,(n_k)}) .$$

But from (*) we have

$$h_j(y_1,\dots,y_m) = h_j(a_1,\dots,a_m) + e_j \quad ,$$

where e_j contains summands which have at least one element of the form f_i as in (*). Then

$$(\dagger) \qquad \sigma(g) = \sum_j \alpha_j h_j(a_1,\dots,a_m) + \sum_j e_j \quad .$$

Let $\{h_1,\dots,h_r\}$ be those h_j used in (†) whose leading terms (with respect to the basis B) have a minimal $C_{n_1,\dots,n_q}$-length. Then $\sigma(g) = 0$ implies that

$$\sum_{j=1}^{r} \alpha_j h_j(a_1,\dots,a_m) = 0$$

This contradicts the linear independence of the basis elements of $A / A_{(s),(n_{q+2}),\dots,(n_k)}$. Thus $\sigma(g) \neq 0$ and

$$\text{Kernel }(\sigma) = A_{(s),(n_{q+2}),\dots,(n_k)} \quad .$$

$$\text{Therefore } \quad S \cong A / A_{(s),(n_{q+2}),\dots,(n_k)} \quad .$$

Note that the converse of the previous theorem is not necessarily true as the following example shows :

<u>Example 3.1</u> : Let $L = F / F_{(3),(9),(3)}$ and S be a subalgebra of L on generators $Y = \{y_1, y_2\}$, where $y_1 \in H_2^{C_3}$ and $y_2 \in H_5^{C_3}$. We have $n_2 = 3$ and if we take $s = 3$ then $K = \{3,4\}$. Then S is free polynilpotent relative to $\{3,3\}$ but $Y \not\subseteq (H_3^{C_3} \cup H_4^{C_3})$.

For a given positive integer $s > 2$, suppose the set $K = \{k_1, \ldots, k_u\}$ is defined as in Definition 3.1 relative to the integer n_{q+2}. Now for $r \geqslant 1$, $r < k_1$, let us assume that the set $T_r = \{t_1, \ldots, t_n\}$ is defined as in Definition 2.4. In this case T_r is completely determined by the integers n_{q+1}, s and r.

<u>Lemma 3.6</u> : Let L, s, K, T_r be as defined above and suppose that $T = \{t_1, \ldots, t_i\}$ is a subset of T_r such that $t_1, \ldots, t_i \leqslant k_u$. If $y \in H_{k_1 - r}^{Cn_1, \ldots, n_q}$, then the set

$$D = \{y\} \cup (H_{t_1}^{Cn_1, \ldots, n_q} \cup \ldots \cup H_{t_i}^{Cn_1, \ldots, n_q})$$

generates a free polynilpotent Lie subalgebra of L relative to the sequence $\{s, n_{q+2}, \ldots, n_k\}$ and one can choose a basis $\underline{B}$ for this subalgebra such that $\underline{B} \subseteq B$.

<u>Proof</u> : D is a subset of B and suppose it is given an order which coincides with that of B. Let $\underline{H}$ be a Hall basis constructed on D and N be the subalgebra of F generated by D. By Lemma 2.10, D freely generates a free nilpotent Lie algebra of class s in $L / L_{(n_1),\ldots,(n_{q+1})}$. Then $N_{(s)} \subseteq L_{(n_1),\ldots,(n_{q+1})}$ and we have

(*) $\quad N_{(s),(n_{q+2}),\ldots,(n_k)} \subseteq L_{(n_1),\ldots,(n_k)} = \{0\}$

Also, we conclude from the order we give to $\underline{H}$ that

$$\underline{H}_1 \cup \ldots \cup \underline{H}_{s-1} \subseteq B_{q+1} \quad ,$$

where

$$B_{q+1} = H_1^{Cn_1,\ldots,n_q} \cup \ldots \cup H_{n_{q+1}-1}^{Cn_1,\ldots,n_q}$$

is a subset of B. Similarly one constructs a basis $\underline{B}$ for $N / N_{(s),(n_{q+2}),\ldots,(n_k)}$ such that the orders on $\underline{B}$ and B coincide. Hence the elements of $\underline{B}$ are non-zero and linearly independent. Together with (*) this implies that D generates a free polynilpotent Lie algebra relative to the sequence $\{s,n_{q+2},\ldots,n_k\}$ in L.

<u>Theorem 3.8</u> : Let L, s, K, T_r and T be as defined before. Then,

(i) if S is a subalgebra of L possessing a generating set Y, where there is only one element $y \in Y$ belomging to $L_{(n_1),\ldots,(n_q),(k_1-r)}$, $y \notin L_{(n_1),\ldots,(n_q),(k_1-r+1)}$ and $Y - \{y\} \subseteq L_{(n_1),\ldots,(n_q),(t_1)}$ and linearly independent modulo $L_{(n_1),\ldots,(n_q),(t_1+1)}$, then S is free polynilpotent relative to the sequence $\{s,n_{q+2},\ldots,n_k\}$ and Y is a set of free generators for S.

(ii) if S is a two-generator subalgebra of L on $Y = \{y_1,y_2\}$ such that $y_1 \in L_{(n_1),\ldots,(n_q),(k_1-r)}$, $y_1 \notin L_{(n_1),\ldots,(n_q),(k_1-r+1)}$ and $y_2 \in L_{(n_1),\ldots,(n_q),(t_1)}$,

$y_2 \notin L_{(n_1),\ldots,(n_q),(t_n+1)}$, then S is free polynilpotent relative to the sequence $\{s, n_{q+2}, \ldots, n_k\}$.

<u>Proof</u> : (i) Take $y \in H^{Cn_1,\ldots,n_q}_{k_1-r}$ and let $Y = \{y, y_1, \ldots, y_m\}$ be a set of generators for S such that

$$y_i = \sum_j \alpha_j d_j + f_i$$

for $i = 1,\ldots,m$, where $\alpha_j \in \underline{k}$, $d_j \in D^{-\{y\}}$, $f_i \in L_{(n_1),\ldots,(n_q),(t_i+1)}$.

(D and N are as in the previous lemma) Let

$$a_i = \sum_j \alpha_j d_j$$

so that

(*) $$y_i = a_i + f_i \ .$$

Then, $\{a_1,\ldots,a_m,y\} \subseteq N$ and it is linearly independent modulo $N_{(2)}$. Thus it freely generates a free polynilpotent Lie algebra relative to the sequence $\{s, n_{q+2}, \ldots, n_k\}$. Let A be the subalgebra of F generated by $\{a_1,\ldots,a_m,y\}$. We now show that $S \cong A / A_{(s),(n_{q+2}),\ldots,(n_k)}$.

Let $\sigma : A \to S$ be the map defined by

$$\sigma : a_i \to y_i \quad \text{for } i = 1,\ldots,m$$
$$y \to y \ .$$

Then, $\sigma (A_{(s),(n_{q+2}),\ldots,(n_k)}) = \{0\}$ and

$$A_{(s),(n_{q+2}),\ldots,(n_k)} \subseteq \text{Kernel } (\sigma) \ .$$

Suppose that there exists a non-zero element g in

$A / A_{(s),(n_{q+2}),\dots,(n_k)}$ such that $\sigma(g) = 0$. We can express g in the form

$$g = \sum_j \alpha_j h_j(a_1,\dots,a_m,y) \quad (\text{ modulo } A_{(s),(n_{q+2}),\dots,(n_k)}) \quad ,$$

where $\alpha_j \in \underline{k}$, $h_j \in \underline{B}$ ($\underline{B}$ is a basis for $A / A_{(s),\dots,(n_k)}$) and not all α_j are zero. Then

$$\begin{aligned} \sigma(g) &= \sum \alpha_j h_j(y_1,\dots,y_m,y) \\ &= \sum_j \alpha_j h_j(a_1,\dots,a_m,y) + \sum_j e_j \quad , \end{aligned}$$

where e_j uses at least one element of the form f_i as in (*). Let $\{h_1,\dots,h_r\}$ be those h_j in (†) whose leading terms have a minimal $C_{m_1,\dots,n_q}$-length. Then every other term in (†) has greater $C_{m_1,\dots,n_q}$-length. Thus $\sigma(g) = 0$ implies that

$$\sum_{i=1}^{r} \alpha_i h_i(a_1,\dots,a_m,y) = 0 \quad .$$

This contradicts the linear independence of the basis elements of $A / A_{(s),(n_{q+2}),\dots,(n_k)}$. Hence $\sigma(g) \neq 0$ and

$$\text{Kernel } (\sigma) = A_{(s),(n_{q+2}),\dots,(n_k)}$$

Therefore S is isomorphic to $A / A_{(s),(n_{q+2}),\dots,(n_k)}$. This proves (i).

(ii) Take $y_1 \in H_{k_1-r}^{Cn_1,\dots,n_q}$ and let

$$y_2 = \sum_j \alpha_j h_j + b \quad ,$$

where $h_j \in (H_{t_1}^{Cn_1,\dots,n_q} \cup \dots \cup H_{t_n}^{Cn_1,\dots,n_q})$, $b \in L_{(n_1),\dots,(n_q),(t_n+1)}$
Then, $Y = \{y_1, y_2\}$ generates a free nilpotent Lie algebra of class s in $L / L_{(n_1),\dots,(n_k)}$. Obviously $S_{(s),(n_{q+2}),\dots,(n_k)}$ is zero and if f is a non-zero element in $S / S_{(s),\dots,(n_k)}$, then $f \neq 0$ in L.

The results corresponding to $s = 2$, established in Chapter 2 for the nilpotent case, can be translated into the polynilpotent case in the usual manner. In particular, if L is a free polynilpotent Lie algebra relative to the sequence $\{n_1,\dots,n_k\}$ and S is an abelian subalgebra of L, then S is free abelian. Thus abelian and free abelian subalgebras of L coincide. Note that an abelian subalgebra of L is necessarily contained in $L_{(n_1),\dots,(n_{k-1})}$.

For a given s suppose that the set $K = \{k_1,\dots,k_u\}$ is defined relative to the integer n_{q+2}. Now for $b \geq 1$, $b < n_{q+2}-k_u$ let us assume that the sets $W_b = \{w_1,\dots,w_e\}$ are defined as in Definition 2.5. In this case W_b is completely determined by the integers n_{q+2}, s and b.

<u>Lemma 3.7</u> : Let L, s, K, W_b be as defined above. Let $W = \{w_j,\dots,w_e\} \subseteq W_b$ such that $w_j,\dots,w_e \geq k_1$. If $y \in H_{k_u+b}^{Cn_1,\dots,n_q}$ then the set

$$D = (H_{w_j}^{Cn_1,\dots,n_q} \cup \dots \cup H_{w_e}^{Cn_1,\dots,n_q}) \cup \{y\}$$

generates a free polynilpotent Lie algebra relative to the sequence $\{s, n_{q+2},\dots,n_k\}$ in L and one can construct a basis $\underline{B}$

for this subalgebra such that $\underline{B} \subseteq B$.

Proof : $D \subseteq B$ and we give D an order which coincides with that of B. Let $\underline{H}$ be a Hall set constructed on D and N be the subalgebra of F generated by D. By Lemma 2.14, D generates a free nilpotent Lie algebra of class s in $L / L_{(n_1),\dots,(n_{q+1})}$. Then

$$N_{(s)} \subseteq L_{(n_1),\dots,(n_{q+1})}$$

which implies that

$$N_{(s),(n_{q+2}),\dots,(n_k)} \subseteq L_{(n_1),\dots,(n_k)} = \{0\}$$

We know that by the order we give to $\underline{H}_1 \cup \dots \cup \underline{H}_{s-1}$ we have

$$\underline{H}_1 \cup \dots \cup \underline{H}_{s-1} \subseteq B_{q+1} \quad ,$$

where

$$B_{q+1} = (H_1^{Cn_1,\dots,n_q} \cup \dots \cup H_{n_{q+1}-1}^{Cn_1,\dots,n_q}) .$$

Proceeding this way one comstructs a basis $\underline{B}$ for the quotient algebra $N / N_{(s),(n_{q+2}),\dots,(n_k)}$ such that $\underline{B} \subseteq B$ as in the proof of Lemma 3.5.

Therefore D generates a free polynilpotent subalgebra in L relative to the sequence $\{s, n_{q+2}, \dots, n_k\}$.

Theorem 3.9 : Let L, K, s, W_b and $W = \{w_j, \dots, w_e\}$ be as defined before. Then

(i) if S is a subalgebra of L which possesses a generating set Y such that only one element $y \in Y$ belongs to $L_{(n_1),\dots,(n_q),(k_u+b)}$, $y \notin L_{(n_1),\dots,(n_q),(k_u+b+1)}$ and

$Y - \{y\} \subseteq L_{(n_1),\dots,(n_q),(w_j)}$ and linearly independent modulo $L_{(n_1),\dots,(n_q),(w_e+1)}$, then S is free polynilpotent relative to the sequence $\{s, n_{q+2}, \dots, n_k\}$ and Y is a set of free generators for S.

(ii) if S is a two-generator subalgebra on the set $Y = \{y_1, y_2\}$ such that $y_1 \in L_{(n_1),\dots,(n_q),(w_1)}$, $y_1 \notin L_{(n_1),\dots,(n_q),(w_e+1)}$ and $y_2 \in L_{(n_1),\dots,(n_q),(k_u+b)}$, $y_2 \notin L_{(n_1),\dots,(n_q),(k_u+b+1)}$, then S is free polynilpotent relative to the sequence $\{s, n_{q+2}, \dots, n_k\}$.

<u>Proof</u> : (i) Take $y \in H^{Cn_1,\dots,n_q}_{k_u+b}$ and let $Y = \{y_1, \dots, y_m, y\}$ be a generating set for S, where

$$y_i = \sum_j \alpha_j d_j + f_i$$

for $i = 1,\dots,m$, $\alpha_j \in \underline{k}$, $d_j \in D^{-\{y\}}$ and $f_i \in L_{(n_1),\dots,(n_q),(w_e+1)}$. (D and N are as in the proof of the previous lemma). Put

$$a_i = \sum_j \alpha_j d_j$$

so that

(*) $$y_i = a_i + f_i \quad .$$

for $i = 1,\dots,m$. The set $\{a_1, \dots, a_m, y\} \subseteq N$ and it is linearly independent modulo $N_{(2)}$. Hence it generates a free polynilpotent Lie algebra relative to the sequence $\{s, n_{q+2}, \dots, n_k\}$. Let A be the subalgebra of F generated by $\{a_1, \dots, a_m, y\}$. Then by an arguement similar to that in the proof of Theorem 3.8 we have

$$S \cong A / A_{(s),(n_{q+2}),\dots,(n_k)} \quad .$$

(ii) The proof of this part is similar to that of Theorem 3.8 part (ii).

For given positive integers $b, r \geqslant 1$, suppose that W_b and T_r are as defined above. Let

$$W_b \cap T_r = \{ v_1, \dots, v_m \}$$

Lemma 3.8 : Let L, s, K, b, r be as stated before and suppose that $W_b \cap T_r = \{ v_1, \dots, v_m \}$ is non-empty. Let

$$D = \{y\} \cup (H_{v_1}^{Cn_1,\dots,n_q} \cup \dots \cup H_{v_m}^{Cn_1,\dots,n_q}) \cup \{ z \}$$

where $y \in H_{k_1 - r}^{Cn_1,\dots,n_q}$ and $z \in H_{k_u + b}^{Cn_1,\dots,n_q}$. Then, D generates a free polynilpotent Lie algebra relative to the sequence $\{ s, n_{q+2}, \dots, n_k \}$ in L and one can construct a basis $\underline{B}$ for this subalgebra such that $\underline{B} \subseteq B$.

Proof : $D \subseteq B$ and we give it an order which coincides with that of B. Let $\underline{H}$ be a Hall set constructed on D and N be the subalgebra of F generated by D. By Lemma 2.17 D generates a free nilpotent Lie algebra of class s in $L / L_{(n_1),\dots,(n_{q+1})}$. Hence

$$N_{(s),(n_{q+2}),\dots,(n_k)} \subseteq L_{(n_1),\dots,(n_k)} = \{ 0 \}$$

An arguement similar to that in the proof of Lemma 3.6 proves that D generates a free polynilpotent Lie algebra relative to $\{ s, n_{q+2}, \dots, n_k \}$.

Theorem 3.10 : Let L, s, K, W_b T_r be as in the previous lemma. If S is a subalgebra of L possessing a generating set Y such that there exists $y \in Y$, $y \in L_{(n_1),\dots,(n_q),(k_1-r)}$, $y \notin L_{(n_1),\dots,(n_q),(k_1-r+1)}$, $z \in Y$, $z \in L_{(n_1),\dots,(n_q),(k_u+b)}$, $z \notin L_{(n_1),\dots,(n_q),(k_u+b+1)}$ and $Y - \{y,z\} \subseteq L_{(n_1),\dots,(n_q),(v_1)}$ and linearly independent modulo $L_{(n_1-,\dots,(n_q),(v_m+1)}$, then S is free polynilpotent relative to the sequence

$$\{s, n_{q+2}, \dots, n_k\}$$

and Y is a set of free generators for S.

<u>Proof</u> : The proof is very similar to that of Theorems 3.8 and 3.9.

If S is a free polynilpotent subalgebra of L relative to the sequence $\{s, n_{q+2}, \dots, n_k\}$, $q < k$, $s < n_{q+1}$, then S need not have a generating set Y which satisfies the hypothesis of one of the theorems stated above. For two-generator subalgebras, however, the classification is complete.

<u>Theorem 3.11</u> : Let $L = F / F_{(n_1),\dots,(n_k)}$ and S be a two-geberator subalgebra of L. Then, S is free poly-nilpotent relative to the sequence $\{s, n_{q+2}, \dots, n_k\}$, $s < n_{q+1}$, $q < k$, if and only if S satisfies the hypothesis of one or more of the theorems stated in this section.

<u>Proof</u> : We have already proved that the condition is necessary. The sufficiency of the condition follows from Theorem 2.10 . If S does not satisfy the hypothesis of one

of the theorems stated above, then S is not free nilpotent of class s in $L / L_{(n_1),\dots,(n_{q+1})}$. Then it is not free polynilpotent relative to $\{s, n_{q+2}, \dots, n_k\}$. This is a contradiction.

For more general subalgebras we have the following result :

Theorem 3.12 : Let S be a free polynilpotent subalgebra of L relative to the sequence $\{s, n_{q+2}, \dots, n_k\}$, $s < n_{q+1}$, $q < k$. Then S has a generating set $Y \subseteq L_{(n_1),\dots,(n_q)}$ such that if $y \in Y$, then $y \notin L_{(n_1),\dots,(n_{q+1})}$.

Proof : Suppose that $Y \nsubseteq L_{(n_1),\dots,(n_q)}$. Then there exists an element $z \in Y$ such that $z \notin L_{(n_1),\dots,(n_q)}$. Let y be any other element of Y and A be the two-generator subalgebra of S generated by $\{y,z\}$. Then A is free polynilpotent relative to the sequence $\{s, n_{q+2}, \dots, n_k\}$. By the previous theorem, $\{y,z\}$ must satisfy the hypothesis of one or more of the theorems mentioned above. Then $z \notin L_{(n_1),\dots,(n_q)}$ leads to a contradiction.

Now suppose that $Y \subseteq L_{(n_1),\dots,(n_q)}$ and there exists an element $y \in Y$ such that $y \in L_{(n_1),\dots,(n_{q+1})}$. Then a similar contradiction arises.

Conjecture : Let $L = F / F_{(n_1),\dots,(n_k)}$ and S be a subalgebra of L. If S is a free polynilpotent Lie algebra with a free generating set Y, then there exists an integer

$q < k$ such that Y freely generates a free nilpotent subalgebra modulo $L_{(n_1),\ldots,(n_{q+1})}$. The converse of this statement is also true.

<u>Conjecture</u> : Let S be a free polynilpotent subalgebra of L relative to the sequence $\{m_1,\ldots,m_e\}$. Then $e \leqslant k$ and if $k-e = p$, we have

$$\begin{aligned}
m_e &= n_k \\
m_{e-1} &= n_{k-1} \\
&\vdots \\
m_2 &= n_{k-(e-2)} = n_{p+2} \\
m_1 &\leqslant n_{p+1} .
\end{aligned}$$

All the theorems proved in this chapter applies to free soluble Lie algebras in a simplified form by putting

$$n_1 = n_2 = \ldots = n_k = 2 .$$

Chapter 4

ON IDEALS OF FREE NILPOTENT AND POLYNILPOTENT LIE ALGEBRAS

In the first section of this chapter we consider ideals in free, free nilpotent and free soluble Lie algebras, which are finitely-generated as subalgebras. The next section deals with the same problem in the context of free polynilpotent Lie algebras. In sections three and four we study the quotient Lie algebra of a lower central term of a free (free polynilpotent) Lie algebra by one of its ideals. In the last section we consider ideals as free subalgebras in free nilpotent (polynilpotent) Lie algebras.

§ 1. Ideals of Free (Free Nilpotent, Free Soluble) Lie Algebras Which Are Finitely-Generated as Subalgebras.

O. Schreier proved in [16] that in a free group a non-trivial finitely-generated subgroup which is normal has finite index. The same question was raised by A. I. Smel'kin in the context of free soluble groups. A necessary and sufficient condition for a [redacted] normal subgroup of a free soluble group to be finitely generated is given by D. I. Eidel'kind in [6] and it is generalized to include the free polynilpotent case.

An analogue of O. Schreier's result for free groups was proved for Lie algebras by B. Baumslag in [2] :

<u>Theorem 4.1</u> : Let F be a free Lie algebra over a field $\underline{k}$ on free generators X. Let S be an ideal of F which is finitely-generated as a subalgebra. Then, $S = \{0\}$ or $S = F$.

A similar result for free metabelian Lie algebras is also given in [2] :

<u>Theorem 4.2</u> : The only possible ideals which are finitely generated as subalgebras of a free metabelian Lie algebra are the zero ideal or the whole algebra.

<u>Corollary 4.1</u> : The theorem mentioned above is also true in the case of free soluble Lie algebras.

If $L = F / F_{(p)}$ is a free nilpotent Lie algebra of class p, then L may have non-trivial ideals which are finitely-generated as subalgebras as in the following example :

<u>Example 4.1</u> : Let $L = F / F_{(4)}$ and suppose that F is the free Lie algebra on the finite generating set X. L has basis $(X \cup H_2 \cup H_3)$, where H is a Hall basis for F constructed on X. Then, $F_{(2)}$ is a finitely-generated subalgebra which is an ideal of L. $F_{(2)}$ is generated by $(H_2 \cup H_3)$ modulo $F_{(4)}$.

<u>Lemma 4.1</u> : Let $L = F / F_{(p)}$ be a free nilpotent Lie algebra on free generators X, $p \geq 2$, and suppose that S is an ideal of L, which is finitely generated by the set $Y \neq \phi$ as a subalgebra. If there is a letter $x \in X$ which does not occur in any of the elements of Y, then $S \subseteq L_{(p-1)}$.

<u>Proof</u> : Let x be as described above and y be any element

of Y. Consider the element $f = xy \in S$. Since f contains the letter x, it cannot be expressed in terms of the generators Y of S. But unless $Y \subseteq L_{(p-1)}$, $f \neq 0$ in L. Therefore, $S \subseteq L_{(p-1)}$.

In particular, we have the following result if L has infinite rank :

Theorem 4.3 : Let $L = F / F_{(p)}$ be a free nilpotent Lie algebra of class $p \geq 2$ on an infinite generating set X and S be an ideal of L, which is finitely generated as a subalgebra. Then, unless $S \subseteq L_{(p-1)}$, S is the zero ideal.

Proof : Let Y be a finite generating set for S. Then the elements of Y uses only a finite number of lettres, say $\{x_1, \ldots, x_k\}$ of X. Hence, there exists $x \in X$ which does not occur in any element of Y. By the previous lemma, unless $S \subseteq L_{(p-1)}$, S is the zero ideal.

§ 2. Finitely Generated Subalgebras of a Free Polynilpotent Lie Algebra Which Are Ideals.

In this section we take $L = F / F_{(n_1),\ldots,(n_k)}$, where $k \geq 2$, $n_i \geq 2$ for $i = 1,\ldots,k.$, and F is a free Lie algebra on free generators X such that $|X| \geq 2$.

Lemma 4.2 : Let $L = F / F_{(n_1),\ldots,(n_k)}$ and suppose that $y \in L_{(n_1),\ldots,(n_q),(e)}$, $y \notin L_{(n_1),\ldots,(n_k),(e+1)}$ and $x \in L_{(n_1),\ldots,(n_p)}$, $x \notin L_{(n_1),\ldots,(n_{p+1})}$, where $e < n_{q+1}$, $p < q < k$. Then, for any positive integer $r \geq 1$, the element

$$\underbrace{x(x(\quad(xy))}_{r\text{-times}}\ldots)$$

belongs to $L_{(n_1),\ldots,(n_q),(e)}$ but not $L_{(n_1),\ldots,(n_q),(e+1)}$.

Proof : Let $h = \mathrm{ld}\ (x)$ and $h' = \mathrm{ld}\ (y)$. (With respect to the basis B of L). Then

$$h \in (H_1^{Cn_1,\ldots,n_p} \cup \ldots \cup H_{n_{p+1}-1}^{Cn_1,\ldots,n_p})$$

$$h' \in H_e^{Cn_1,\ldots,n_q} .$$

But $p < q$ implies that

$$\mathrm{ld}\ (xy) \in L_{(n_1),\ldots,(n_q),(e)}$$

Since $C_{n_1,\ldots,n_q}$-length $(h) = 0$, we have

$$C_{n_1,\ldots,n_q}\text{-length } (\mathrm{ld}\ (xy)) = e .$$

(Note that hh' need not belong to the basis B). Hence

$$xy \notin L_{(n_1),\ldots,(n_q),(e+1)} .$$

Similarly one can prove by induction on r that

$$\underbrace{x(x(\quad(xy))}_{r\text{-times}}\ldots) \in L_{(n_1),\ldots,(n_q),(e)}$$

but

$$\underbrace{x(x(\ldots(xy))}_{r\text{-times}}\ldots) \notin L_{(n_1),\ldots,(n_q),(e+1)} .$$

Lemma 4.3 : Let $L = F / F_{(n_1),\ldots,(n_k)}$ and suppose that $k \geq 2$. Then centre of L is zero.

Proof : Centre of L is the set of elements in L whose products with every other element of L is zero. Let $C(L)$ denote the centre of L and suppose that $C(L) \neq \{0\}$. Then, there is $y \neq 0$ in $C(L)$ and suppose that $y \in L_{(n_1),\dots,(n_q),(e)}$ but $y \notin L_{(n_1),\dots,(n_q),(e+1)}$ for some $q < k$, $e < n_{q+1}$. Let X be a free generating set for L and $x \in X$. By the previous lemma, we have $xy \notin L_{(n_1),\dots,(n_q),(e+1)}$. Thus $xy \neq 0$ in L, which contradicts y belonging to $C(L)$.

Therefore, $C(L) = \{0\}$.

Theorem 4.4 : Let $L = F / F_{(n_1),\dots,(n_k)}$, $n_i \geq 2$, $k \geq 2$, and S be an ideal of L. If S is finitely generated as a subalgebra, then $S = \{0\}$ or $S = L$.

Proof : We analyse three cases seperately.

Case I : $S \subseteq L_{(2)}$. Then there exists q, $1 \leq q < k$ and $e < n_{q+1}$ such that $S \subseteq L_{(n_1),\dots,(n_q),(e)}$ and $S \not\subseteq L_{(n_1),\dots,(n_q),(e+1)}$. Let Y be a finite generating set for S. Then there exists an element $y \in Y$ such that

$$y = \alpha h + \dots$$

where $h \in H_e^{Cn_1,\dots,n_q}$. Let $x \in X$, the free generating set for L, and suppose that d is the maximum number of times that x appears in any element of Y. Consider the elements

$$f_r = \underbrace{x(x(\dots(xy))\dots)}_{r\text{-times}} \in S$$

for any $r \geq 1$. Obviously $f_r \in L_{(n_1),\dots,(n_q),(e)}$ but also $f_r \notin L_{(n_1),\dots,(n_q),(e+1)}$ for any $r \geq 1$. Hence f_r cannot be expressed as a product of elements of Y since any such product would belong to $L_{(n_1),\dots,(n_q),(2e)}$. But when $r > d$, f_r can neither be expressed as a linear combination of the elements in Y. This contradiction proves that $S = \{0\}$.

Case II : Suppose that F is of infinite rank, freely generated by the infinite set X. Let S be finitely-generated by Y as a subalgebra. Then there exists an element $x \in X$ which does not appear in any of the elements of Y. Let y be an arbitrary element of Y. Then $xy \in S$ cannot be expressed in terms of the generators Y. Hence we must have $S = \{0\}$.

Case III : Suppose that F is freely generated by the finite set $X = \{x_1,\dots,x_n\}$ and $S \not\subseteq F_{(2)}$. Let $Y = \{y_1,\dots,y_m\}$ be a finite generating set for S. The set Y generates S modulo $F_{(n_1),\dots,(n_k)}$. Then Y (or a subset of Y) generates S modulo $F_{(2),(2)}$, since $F_{(n_1),\dots,(n_k)} \subseteq F_{(2),(2)}$. Obviously S is an ideal of F modulo $F_{(2),(2)}$. If we have

$$S / F_{(2),(2)} \neq F / F_{(2),(2)}$$

then we have a finitely-generated subalgebra of a free metabelian Lie algebra, which is an ideal and which is not equal to the whole algebra. This contradicts Theorem 4.2. Hence we must have $S / F_{(2),(2)} = F / F_{(2),(2)}$. ($S + F_{(2),(2)} = F$). Then $|Y| \geq |X|$ and $m \geq n$. We can now choose a generating set $Z = \{z_1,\dots,z_m\}$ for S modulo $F_{(n_1),\dots,(n_k)}$ such that

$$z_1 = x_1 + f_1$$
$$\vdots$$
$$z_n = x_n + f_n$$
$$z_{n+1} = f_{n+1}$$
$$z_m = f_m \quad ,$$

where $f_1,\dots,f_m \in F_{(2),(2)}$ and thus $Z - \{z_{n+1},\dots,z_m\} \subseteq F_{(2),(2)}$. Let us assume that $S \neq L$. Then there exists $x_i \in X$ such that $x_i \notin S$. Thus we have $z_i \in Z$, $z_i = x_i + f_i$, satisfying $f_i \notin S$. Let $z_1 = x_1 + f_1$ be such an element, $f_1 \notin S$. Suppose first that there is at least one more element, say $z_2 = x_2 + f_2$ with $f_2 \notin S$. Consider

$$x_1 z_2 = x_1 x_2 + x_1 f_1 \in S$$

The only way $x_1 x_2$ can be expressed in terms of the generators Z is by the product $z_1 z_2$. Hence

$$x_1 z_2 = z_1 z_2 + \dots$$
$$= x_1 x_2 + x_1 f_2 + f_1 x_2 + f_1 f_2 \quad .$$

Therefore, $f_1 x_2 + f_1 f_2$ must belong to S. Since $f_1 x_2 \in F_{(2),(2)}$, it cannot be a linear combination of $\{z_1,\dots,z_n\}$ nor a product of $\{z_1,\dots,z_n\}$. The lettre x_2 is only produced by using the generator z_2 in S and then f_1 would have to belong to S, a contradiction. Furthermore, $f_1 x_2$ cannot be a product of $\{z_{n+1},\dots,z_m\}$ either, since any such product cannot start with x_2, an element of X. Hence $f_1 x_2$ must be a linear combination of $\{z_{n+1},\dots,z_m\}$. Now consider

$$x_1(x_1 z_2) = x_1(x_1 x_2) + x_1(x_1 f_2) \in S$$

We have

$$\begin{aligned} x_1(x_1 z_2) &= z_1(z_1 z_2) + \dots \\ &= x_1(x_1 x_2) + x_1(x_1 f_2) + x_1(f_1 x_2) + f_1(x_1 x_2) \\ &\quad x_1(f_1 f_2) + f_1(f_1 f_2) + f_1(x_1 f_2) + f_2(f_1 x_2) \ . \end{aligned}$$

Then, $f_1(x_1x_2) + x_1(f_1x_2)$ must belong to S, together with the other terms in the product above. Similarly one concludes that $f_1(x_1x_2) + x_1(f_1x_2)$ must be a linear combination of $\{z_{n+1}, \dots, z_m\}$.

Now form the element

$$\begin{aligned} f_r &= \underbrace{x_1(x_1(\dots(x_1}_{r\text{-times}} z_1))\dots) \in S \\ &= \underbrace{x_1(x_1(\dots(x_1}_{r\text{-times}} x_2))\dots) + \underbrace{x_1(x_1(\dots(x_1}_{r\text{-times}} f_2))\dots) \end{aligned}$$

Then

$$(\dagger) \qquad \begin{aligned} f_r &= \underbrace{z_1(z_1(\dots(z_1}_{r\text{-times}} z_2))\dots) + \dots \\ &= x_1(x_1(\dots(x_1x_2))\dots) + x_1(x_1(\dots(x_1f_2))\dots) \\ &\quad x_1(x_1(\dots(x_1(f_1x_2))\dots) + x_1(x_1(\dots(f_1(x_1x_2))\dots) \\ &\quad + \dots \end{aligned}$$

Then the element

$$g_r = \underbrace{x_1(x_1(\dots(x_1(f_1}_{r\text{-times}}x_2))\dots) + \underbrace{x_1(x_1(\dots(x_1(f_1(x_1}_{r\text{-times}}x_2))\dots)$$

must belong to S and since it cannot be a product of $\{z_{n+1}, \dots, z_m\}$, it can only be a linear combination of $\{z_{n+1}, \dots, z_m\}$. Let d be the maximum number of times that x_1 appears in any element of $\{z_{n+1}, \dots, z_m\}$. Then, when $r > d$, the element f_r cannot be expressed as a linear combination of the $\{z_{n+1}, \ldots, z_m\}$. If it is written in the form (†), then the element g_r cannot be expressed as a linear combination of the $\{z_{n+1}, \ldots, z_m\}$. This is a contradiction.

Now suppose that $z_1 = x_1 + f_1$ is the only element with $f_1 \notin S$, that is, suppose that $x_2, \ldots, x_n \in S$, $z_1 = x_1, \ldots,$ $z_n = x_n$. Then $f_1 x_2, \ldots, f_1 x_n \in S$. But

$$\begin{aligned} z_1 x_2 &= x_1 x_2 + f_1 x_2 \\ &\vdots \\ z_1 x_n &= x_1 x_n + f_1 x_n \end{aligned}$$

implies that $x_1 x_2, \ldots, x_1 x_n \in S$. Then, H_2, the elements in H (Hall basis for F) of length 2, belong to S. Suppose that $H_2 \cup \ldots \cup H_v$ belong to S. If $h \in H_{v+1}$, $v > 2$, then $h = h_1(h_2 h_3)$ and either $h_1 \in H_2 \cup \ldots \cup H_v$ or $h_2 h_3 \in H_2 \cup \ldots \cup H_v$. Thus $H_{v+1} \subseteq S$ and

$$\bigcup_{j=2}^{\infty} H_j \subseteq S .$$

Then, if B is the basis for $L = F / F_{(n_1), \ldots, (n_k)}$, we have $B - \{x_1\} \subseteq S$. But f_1 can be expressed as a linear combination of elements from $B - \{x_1\}$. Hence $f_1 \in S$ and $x_1 \in S$, which implies that $S = L$.

Therefore in any case we conclude that $S = L$.

As in the case of absolutely free Lie algebras, there is an analogue of Theorem 4.4 for groups which is a little different. D. I. Eidel'kind has proved the following in [6] :

" In the free polynilpotent group G of finite rank, corresponding to the sequence $\{n_1, \ldots, n_k\}$, $k > 1$, a normal subgroup N, which is finitely-generated as a subgroup, satisfies $N \neq 1$ if and only if its index is finite modulo $G_{(n_1), \ldots, (n_{k-1})}$."

§ 3. The Quotient Lie Algebra of a Term of the Lower Central Series of a Free Lie Algebra by One of Its Ideals

Let F be a free group of finite rank and S be a normal subgroup of F such that S has finite index in F. D. Spellman studied the quotient group $F_{(m)} / S_{(m)}$ in [21]. Schreier's theorem states that in a free group, a normal subgroup which is finitely-generated has finite index. Hence in the problem mentioned above one can consider subgroups of infinite or finite rank. However, as we have stated in § 1, in a free Lie algebra a finitely generated subalgebra which is an ideal is trivial or the whole algebra.

In this section we consider Lie algebra analogues of the problem mentioned above for the non-trivial case of ideals which are of infinite rank as subalgebras.

Let F be a free Lie algebra on a free generating set X, $|X| \geqslant 2$. Let S be an ideal in F and suppose that it is not finitely generated as a subalgebra. Assume that Y is a free generating set for S. Obviously $S_{(m)} \subseteq F_{(m)}$ and $S_{(m)}$ is a characteristic ideal in S. Then $S_{(m)}$ is an ideal of F and thus $S_{(m)} \triangleleft F_{(m)}$. Hence we can form the quotient Lie algebra $F_{(m)} / S_{(m)}$, $m \geqslant 1$.

Theorem 4.5 : Let S be a non-trivial ideal of F and suppose that $S \subseteq F_{(2)}$. Then $F_{(m)} / S_{(m)}$ has infinite rank for $m \geq 2$.

Proof : Let X be a free generating set for F, $|X| \geq 2$, and suppose that it is given a total order. Take $x_1, x_2 \in X$ such that $x_1 < x_2$. Form the set A, for $m \geq 2$, defined by

$$A = \left\{ f_r = \underbrace{x_1(x_1(\ldots(x_1}_{r\text{-times}}x_2))\ldots) \ : \ r \geq m-1 \right\}$$

Then $A \subseteq C_2$, the free generating set for $F_{(2)}$. Now, $S \subseteq F_{(2)}$ implies that $S_{(2)} \subseteq F_{(2),(2)}$. But since $C_2 \cap F_{(2),(2)} = \phi$ we conclude that $A \cap S_{(2)} = \phi$. Furthermore, A is an infinite subset of C_m, the free generating set for $F_{(m)}$, and $A \cap S_{(2)} = \phi$ implies that $A \cap S_{(m)} = \phi$, since $S_{(m)} \subseteq S_{(2)}$. Also since the elements of A are distinct, the cosets

$$[a] = a + S_{(m)} ,$$

where $a \in A$, are distinct. Futhermore the cosets $[a]$, $a \in A$, cannot be expressed in terms of a finite number of generators for $F_{(m)} / S_{(m)}$.

Therefore $F_{(m)} / S_{(m)}$ has infinite rank, for $m \geq 2$.

Lemma 4.5 : Let S be a non-trivial ideal of F such that $S \not\subseteq F_{(2)}$. Suppose that we can choose free generating sets X and Y for F and S respectively such that X has an element x which does not appear in the homogeneous component of first degree in any $y \in Y$. Then $F_{(m)} / S_{(m)}$ has infinite rank for $m \geq 2$.

Proof : Let X and Y be as described above. Since

$S \not\subseteq F_{(2)}$ there exists $y \in Y$ such that

$$y = \sum_j \alpha_j x_j + f ,$$

where $x_j \in X$, $x_j \neq x$, $f \in F_{(2)}$. Consider the set A defined by

$$A = \Big\{ f_r = \underbrace{x(x(..(x}_{r\text{-times}}y))..) \ : \ r \geq m-1 \Big\}$$

Obviously $A \subseteq F_{(m)}$ and $A \subseteq S$. If $f_r \in A$, then

$$f_r = \sum_j \alpha_j \underbrace{x(x(..(x}_{r\text{-times}}x_j))..) + f'_r ,$$

where $f'_r \in F_{(r+2)}$. Since x does not appear in the homogeneous component of first degree in any $y \in Y$, the term $\underbrace{x(x(..(x}_{r\text{-times}}x_j))..)$ cannot be expressed as a (linear combination of) products of elements in Y. Then $\underbrace{x(x(..(x}_{r\text{-times}}x_j))..)$ is a linear combination the elements in Y. This implies that $f_r \notin S_{(2)}$ for any $r \geq 1$. Hence $A \cap S_{(2)} = \phi$ and thus $A \cap S_{(m)} = \phi$. Also since every element of A has leading term with a different X-length the elements of A are distinct and thus the cosets

$$[a] = a + S_{(m)}$$

are distinct. Furthermore the cosets $[a]$, $a \in A$, cannot be expressed in terms of a finite number of generators for $F_{(m)} / S_{(n)}$, since the same for the homogeneous component of the lowest X-lengths in $f_r \in A$.

Therefore $F_{(m)} / S_{(m)}$ has infinite rank for $m \geq 2$.

<u>Corollary 4.2</u> : Let S be an ideal of F such that $S \nsubseteq F_{(2)}$. If there is a free generating set X for F such that $X = X_1 \cup X_2$, where $\phi \neq X_2 \subseteq S$ and X_1 is linearly independent modulo S, then $F_{(m)} / S_{(m)}$ has infinite rank for $m \geqslant 2$.

<u>Proof</u> : The result follows from the previous lemma.

<u>Example 4.2</u> : Let F be a free Lie algebra on free generators $X = \{a,b,c,d\}$ and S be the ideal of F generated by $\{b,c,d\} \cup F_{(2)}$. Then, F / S is one-dimensional free abelian Lie algebra generated by the coset $a + S$. However, $F_{(2)} / S_{(2)}$ has infinite rank since the set A defined by

$$A = \left\{ f_r = \underbrace{a(a(..(a}_{r\text{-times}}c))..) \quad : \quad r \geqslant 1 \right\}$$

satisfies $A \cap S_{(2)} = \phi$ and A is a subset of the free generating set C_2 of $F_{(2)}$. In fact, $F_{(2)} / S_{(2)}$ is free abelian and for any $m \geqslant 2$, $F_{(m)} / S_{(2)} \cap F_{(m)}$ has also infinite rank.

<u>Corollary 4.3</u> : Let S be an ideal of a finitely-generated free Lie algebra F such that $F_{(m)} / S_{(m)}$ has infinite rank. Then, $F_{(n)} / S_{(m)} \cap F_{(n)}$ has infinite rank for $n \geqslant m \geqslant 2$.

<u>Proof</u> : Let A' be an infinite free generating set for $F_{(m)} / S_{(m)}$, where

$$A' = \left\{ a + S_{(m)} \quad : \quad a \in A \right\}$$

for some infinite set A. Since F is finitely-generated, $A \cap F_{(n)}$ must be an infinite set for any $n \geqslant m$ and

$$(A \cap F_{(n)}) \cap S_{(m)} = \phi$$

since $A \cap S_{(m)} = \phi$. Thus $F_{(m)} / S_{(m)} \cap F_{(n)}$ has infinite rank for $n \geqslant m \geqslant 2$.

Corollary 4.4 : Let $S \subsetneq F_{(2)}$ be a non-trivial ideal of F. Then $\delta^k F / \delta^k S$ has infinite rank for $k \geqslant 1$.

Proof : If $k = 1$, then $\delta^1 F / \delta^1 S = F_{(2)} / S_{(2)}$ has infinite rank by Theorem 4.5 . If $k > 1$, repeated use of the same theorem proves the result.

Corollary 4.5 : Let $S \subsetneq F_{(2)}$ be a non-trivial ideal of F. Then $F_{\underbrace{(m),\ldots,(m)}_{k\text{-times}}} / S_{\underbrace{(m),\ldots,(m)}_{k\text{-times}}}$ has infinite rank for $r \geqslant k \geqslant 1$, $m \geqslant 2$.

The results proved in this section are likely to be true for any non-zero ideal S of F such that $S \neq F$, although I have been unable to produce a proof for this general case.

Conjecture : Let S be a non-zero ideal of a free Lie algebra F such that $S \neq F$, where rank of $F > 1$. Then, $F_{(m)} / S_{(m)}$ has infinite rank.

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§ 4. The Quotient Lie Algebra of a Lower Central Term of a Free Nilpotent (Polynilpotent) Lie Algebra by One of Its Ideals

If L is a free nilpotent Lie algebra of finite rank, then there are only finitely many linearly independent elements in L and any subalgebra or quotient algebra of L (or of a lower central term of L) is finite dimensional as a module and hence finitely-generated as a Lie algebra. We now consider free nilpotent Lie algebras of infinite rank.

Theorem 4.6 : Let $L = F / F_{(p)}$, $p \geqslant 2$, be a free nilpotent Lie algebra of infinite rank and S be a non-zero ideal in L such that $S \subseteq F_{(2)}$. Then $L_{(m)} / S_{(m)}$ has infinite rank for $1 \leqslant m \leqslant p-1$.

Proof : Let C_m be a free generating set for $F_{(m)}$ and A be the set defined by

$$A = \left\{ f \in C_m : X\text{-length } (f) = m \right\}$$

Since $S \subseteq L_{(2)}$ we have $A \cap S_{(m)} = \phi$. Furthermore, since L has infinite rank, A is an infinite set. This proves the theorem.

Conjecture : Let $L = F / F_{(p)}$ be of infinite rank and S be a non-zero ideal of L such that $S \neq L$. Then, $L_{(m)} / S_{(m)}$ has infinite rank.

Theorem 4.7 : Let $L = F / F_{(n_1),\ldots,(n_k)}$ be a free polynilpotent Lie algebra of rank not equal to one, $k \geqslant 2$, $n_i \geqslant 2$. Let S be an ideal of L such that $S \subseteq F_{(2)}$. Then, $L_{(m)} / S_{(m)}$ has infinite rank for $m \geqslant 2$.

Proof : Since $S \subseteq L_{(2)}$ we have $S_{(2)} \subseteq L_{(2),(2)}$. Hence, if C_2 is the free generating set for $F_{(2)}$, then $C_2 \cap S_{(2)} = \phi$, since $C_2 \cap F_{(2),(2)} = \phi$. Let $A = C_2 \cap F_{(m)}$. Then A is an infinite set and $A \cap S_{(2)} = \phi$ implies that $A \cap S_{(m)} = \phi$. Therefore $L_{(m)} / S_{(m)}$ has infinite rank.

Conjecture : Let L be as in the previous theorem and S be any non-zero ideal of L such that $S \neq L$. Then $L_{(m)} / S_{(m)}$ has infinite rank for $m \geq 2$.

§ 5. Ideals as Free Subalgebras in Free Nilpotent (Polynilpotent) Lie Algebras

An important problem relating to tha ideals in a free nilpotent (polynilpotent) Lie algebra is the following :

Let $L = F / F_{(p)}$, where F is a free Lie algebra. Can an ideal of L be free nilpotent as a subalgebra ?

If S is an abelian ideal of L, then it is free abelian by Theorem 2.7 . Hence the problem is trivial for abelian ideals. In general, if S is a non-abelian ideal of L, it need not be free nilpotent as a subalgebra. (Take $L_{(2)}$ as an ideal of $L = F / F_{(5)}$).

We need first the following :

Lemma 4.6 : Let $L = F / F_{(p)}$ be a free nilpotent Lie algebra of finite rank on free generating set X and Y be a subset of L which is linearly independent modulo $F_{(2)}$. If $|Y| = |X|$, then $\langle Y \rangle$, the subalgebra of L generated by Y, is equal to L. If $|Y| < |X|$, then one can choose a free generating set $T = T_1 \sqcup T_2$ for L such that $T_2 = Y$ and $T_1 \cap \langle Y \rangle = \phi$.

Proof : Suppose that $|Y| = |X| = n$ and $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_n\}$. Then

$$(*) \qquad \begin{aligned} y_1 &= \alpha_{11}x_1 + \dots + \alpha_{1n}x_n + f_1 \\ &\vdots \\ y_n &= \alpha_{n1}x_1 + \dots + \alpha_{nn}x_n + f_n , \end{aligned}$$

where $f_1,\dots,f_n \in F_{(2)}$, $\alpha_{ij} \in \underline{k}$ for $i,j = 1,\dots,n$. Since Y is linearly independent modulo $F_{(2)}$, the elements

$$(\dagger) \qquad \begin{aligned} t_1 &= \alpha_{11}x_1 + \dots + \alpha_{1n}x_n \\ &\vdots \\ t_n &= \alpha_{n1}x_1 + \dots + \alpha_{nn}x_n \end{aligned}$$

form a set of free generators for F. (see [5]). Let $T = \{t_1,\dots,t_n\}$ so that $\langle T\rangle$, the subalgebra of L generated by T, equals L. Let $w(y_1,\dots,y_n)$ be any homogeneous element of Y-length $= p-1$ in $\langle Y\rangle$. Then,

$$w(y_1,\dots,y_n) = w(t_1,\dots,t_n) + g ,$$

where g contains elements of the form f_i as in (*). Then $g \in F_{(p)}$ and thus

$$w(y_1,\dots,y_n) = w(t_1,\dots,t_n) \in Y .$$

Hence any element of $F_{(p-1)}$ in L belongs to $\langle Y\rangle$. In particular, if H is a Hall basis for L on T, then $H_{p-1} \subseteq \langle Y\rangle$. Similarly if $w'(y_1,\dots,y_n)$ is a homogeneous element of Y-length $= p-2$ in $\langle Y\rangle$, then

$$w'(y_1,\dots,y_n) = w'(t_1,\dots,t_n) + g' ,$$

where $g \in F_{(p-1)}$. But by the above remark $g' \in \langle Y\rangle$ so that $w'(t_1,\dots,t_n) \in \langle Y\rangle$, and $H_{p-2} \subseteq \langle Y\rangle$. Similarly one proceeds to show that

$$H_1 \cup \dots \cup H_{p-1} \subseteq \langle Y\rangle$$

which implies that $\langle Y\rangle = \langle T\rangle = L$.

If $|Y| < |X|$, $Y = \{y_1,\dots,y_m\}$, $m < n$, where y_i are as in $(*)$, then the subalgebra of L generated by $T_2 = \{t_1, \ldots, t_m\}$, where t_i are as in $(\dagger)$, equals $\langle Y \rangle$. Obviously one can choose elements $\{t_{m+1}, \ldots, t_n\} = T_1$ such that $T = T_1 \cup T_2$ generates L freely.

Note that an analogue of this lemma is not true for free polynilpotent Lie algebras as the following counterexample shows :

Example 4.3 : Let $X = \{x_1, x_2\}$ and $L = F / F_{(2),(2)}$ be a free metabelian Lie algebra on free generators X. Let $Y = \{y_1, y_2\}$, where

$$y_1 = x_1 + x_1 x_2$$

$$y_2 = x_2$$

Then Y freely generates a free metabelian subalgebra of rank 2 but $\langle Y \rangle \neq L$, since $x_1 \notin \langle Y \rangle$.

Theorem 4.8 : Let $L = F / F_{(p)}$, $p > 2$, be a free nilpotent Lie algebra and S be a non-zero ideal of L, $S \neq L$. Then, S cannot be free nilpotent of class s as a subalgebra, where $2 < s < p$.

Proof : Suppose that S is free nilpotent of class s, $2 < s < p$. Let Y be a free generating set for S. Then there is an integer q, $2 \leq q < p/2$ such that $Y \subseteq F_{(q)}$ but $Y \nsubseteq F_{(q+1)}$. (If $q = 1$, then one can easily get a contradiction) Let $y_1 \in Y$ such that

$$y_1 = \alpha h_1 + \ldots$$

where $\alpha \in \underline{k}$ and $h_1 \in H_q$ (H is the Hall basis for L). Then, $x_1y_1 \in S$ for $x_1 \in X$ and it cannot be expressed as a linear combination of products of elements in Y. Hence it must be a linear combination of the elements of Y. This implies that x_1y_1 is a homogeneous element of first degree in S. But X-length ($\mathrm{ld}\,(x_1y_1)$) $= q+1$ and thus one can construct an element in $S_{(s)}$ whose leading term has X-length equal to $(s-1)\,q + q + 1 = s\,q + 1$. Hence we have

$$(*) \qquad s\,q + 1 \geqslant p\,.$$

Now consider the elements

$$g_1 = \underbrace{x_1(x_1(\ldots(x_1}_{(q-1)\text{-times}}y_1))\ldots) = 9\,\underbrace{x_1(x_1(\ldots(x_1}_{(q-1)\text{-times}}h_1))\ldots) + \ldots$$

$$g_2 = \underbrace{x_2(x_1(\ldots(x_1}_{(q-1)\text{-times}}y_1))\ldots) = 9\,\underbrace{x_2(x_1(\ldots(x_1}_{(q-1)\text{-times}}h_1))\ldots) + \ldots$$

where $x_1, x_2 \in X$. Obviously

$$\begin{aligned} \text{X-length}\,(\,\mathrm{ld}\,(g_1)\,) &= \text{X-length}\,(\,\mathrm{ld}\,(g_2)\,) \\ &= (q-1) + q \\ &= 2q-1 \end{aligned}$$

Since any product of the elements in Y would have X-length greater than or equal to $2q$, g_1 and g_2 cannot be expressed as a products of the y_i's in Y or as a linear combination of such products. Then they are linear combinations of the y_i's themselves, which implies that g_1 and g_2 are homogeneous elements of first degree in S. Then the element v of S defined by

$$v = \underbrace{g_1(g_1(\ldots(g_1}_{(s-2)\text{-times}}g_2))\ldots)$$

belongs to $S_{(s-1)}$ and thus $\mathrm{ld}\,(v) \notin F_{(p)}$ which implies that

$$(s-2)\,(2q-1) + (2q-1) < p$$

Then

$$(s-1)\,(2q-1) < p$$

$$(sq+1) + (sq-2q-s) < p$$

But $(sq+1) \geq p$ by $(*)$ and hence

$$(sq-2q-s) < 0$$

$$(\dagger) \qquad (s-2)\,q < s\,.$$

If $s \geq 4$, then $(\dagger)$ leads to a contradiction since $q \geq 2$. Hence the only case left to consider is $s = 3$.

Suppose now that $s = 3$. Then $(\dagger)$ implies that $q < s$ and hence $q < 3$. Thus $q = 2$. Then there exists an element $y \in Y$ such that

$$y = \xi h + \ldots$$

where $h \in H_2$. Let $h = x_i x_j$, $x_i < x_j$. Then

$$u_1 = x_i h = x_i(x_i x_j) \in S$$

and u_1 can only be expressed as a linear combination of the elements in Y. Similarly

$$u_2 = x_i(x_i h) = \xi\, x_i(x_i(x_i x_j)) + \ldots \in S$$

$$u_3 = x_j(x_i h) = \xi\, x_j(x_i(x_i x_j)) + \ldots \in S$$

and u_2, u_3 cannot be expressed as a linear combination of products of elements in Y. Hence u_1, u_2 and u_3 are homogeneous elements of first degree in S. Thus $u_2 u_3 \in S_{(2)}$, $u_2 u_3 \in S_{(3)}$

and $u_1(u_1u_2) \in S_{(3)} = \{0\}$. Then

$$\text{X-length (ld } (u_2u_3) \text{)} = 4 + 4 < p$$

$$\text{X-length (ld } (u_1(u_1u_2)) \text{)} = 2 + 2 + 3 \geqslant p$$

The last two inequalities imlply that

$$8 < p \leqslant 7$$

which is a contradiction.

Therefore S cannot be free nilpotent of class s, $2 < s < p$.

<u>Theorem 4.9</u> : Let $L = F / F_{(p)}$, $p > 2$, be a finitely-generated free nilpotent Lie algebra and S be an ideal of L. Then S cannot be free nilpotent of class p as a subalgebra, unless $S = L$.

<u>Proof</u> : Suppose that L is finitely-generated by $X = \{x_1, \ldots, x_n\}$ and S, an ideal of L, is free nilpotent of class p. Then S has a free generating set which is linearly independent modulo $F_{(2)}$. Let $Y = \{y_1, \ldots, y_m\}$ be such a free generating set for S,

$$y_i = \alpha_{i1}x_1 + \ldots + \alpha_{in}x_n + f_i ,$$

where $\alpha_{ij} \in \underline{k}$ for $j = 1, \ldots, n$, $i = 1, \ldots, m$ and $f_i \in F_{(2)}$. At least m of the x_i's appear in the y_j's. (Otherwise we contradict Y being linearly independent modulo $F_{(2)}$). If $m = n$. then $S = L$ by Lemma 4.6 . Let us assume that $m < n$. Put

$$z_i = \alpha_{i1}x_1 + \ldots + \alpha_{in}x_n$$

for $i = 1,\ldots,m$. Then $\{z_1,\ldots,z_m\}$ can be completed to a free generating set $Z = \{z_1,\ldots,z_m,z_{m+1},\ldots,z_n\}$ of F. Let

$$v = z_n y_1 = z_n z_1 + z_n f_1 \in S$$

Then v cannot be expressed in terms of (linear combination of) products of Y, since no element in Y starts with z_n. Neither can v be expressed as a linear combination of the $\{y_1,\ldots,y_m\}$ since such a linear combination would not belong to $F_{(2)}$ but $v \in F_{(2)}$. This contradiction proves the result.

We now prove similar results for ideals in polynilpotent Lie algebras.

<u>Theorem 4.10</u> : Let $L = F / F_{(n_1),\ldots,(n_k)}$, $n_i \geq 2$, $k \geq 2$, be a free polynilpotent Lie algebra and S be an ideal of L. Then S cannot be free polynilpotent relative to the sequence $\{s,n_{q+2},\ldots,n_k\}$, where $1 \leq q < k$, $2 < s < n_{q+1}$, $n_q > 2$.

<u>Proof</u> : Let S be an ideal of L and suppose that it is free polynilpotent relative to the sequence $\{s,n_{q+2},\ldots,n_k\}$ as a subalgebra, $1 \leq q < k$, $2 < s < n_{q+1}$, $n_q > 2$. Let Y be a free generating set for S. Then, by Theorem 3.12, $Y \subseteq F_{(n_1),\ldots,(n_q)}$ and no $y \in Y$ belongs to $F_{(n_1),\ldots,(n_{q+1})}$. Let $C_{n_1,\ldots,n_q}$ be the free generating set for $F_{(n_1),\ldots,(n_q)}$ defined in Chapter 1. As in the proof of Theorem 4.8, we must have $Y \subseteq F_{(n_1),\ldots,(n_q),(2)}$, since we require $s \neq n_{q+1}$. Then there exists an integer p, $2 \leq p < n_{q+1}/2$, such that

$Y \subseteq F_{(n_1),\dots,(n_q),(p)}$ but $Y \not\subseteq F_{(n_1),\dots,(n_q),(p+1)}$. Let $y \in Y$ such that

$$y = \beta h + \dots$$

where $\beta \in \underline{k}$ and $h \in H_p^{Cn_1,\dots,n_q}$. Let c_1 be an element of $C_{n_1,\dots,n_q}$. Then

$$v = c_1 y = \beta c_1 h + \dots \in S$$

is an homogeneous element of first degree in S and it has leading term whose $C_{n_1,\dots,n_q}$-length is $p+1$. Then one can construct an element in $S_{(s)}$ whose $C_{n_1,\dots,n_q}$-length is equal to $(s-1)p + (p+1) = sp + 1$. Hence

$$(*) \qquad sp + 1 \geq n_{q+1}$$

Let $c_2 \in C_{n_1,\dots,n_q}$, $c_2 \neq c_1$. Consider the elements

$$g_1 = c_1(c_1(\dots(c_1 y))\dots) = \beta\, c_1(c_1(\dots(c_1 h))\dots) + \dots \in S$$

$$g_2 = \underbrace{c_2(c_1(\dots(c_1}_{(p-1)\text{-times}} y))\dots) = \beta\, \underbrace{c_2(c_1(\dots(c_1}_{(p-1)\text{-times}} h))\dots) + \dots \in S$$

Obviously g_1, g_2 satisfy

$$C_{n_1,\dots,n_q}\text{-length}\ (\ \mathrm{ld}\ (g_i)\ ) = 2p - 1$$

for $i = 1,2$. Since any product of elements of Y would have $C_{n_1,\dots,n_q}$-length $\geq 2p$, g_1 and g_2 cannot be expressed in terms of products of the y_i's. Hence they are linear combinations of the elements of Y and thus g_1, g_2 are homogeneous elements of first degree in S. Let

$$u = \underbrace{g_1(g_1(\dots(g_1}_{(s-2)\text{-times}} g_2))\dots) \in S$$

Then we have

$$(s-2)\ (2p-1) + (2p-1) < n_{q+1}$$

$$(sp+1) + (sp-2p-s) < n_{q+1}$$

But $(sp+1) \geqslant n_{q+1}$ by $(*)$ and thus

$$(\dagger) \qquad (sp - 2p - s) < 0 .$$

If $s \geqslant 4$, $(\dagger)$ lesds to an obvious contradiction. If $s = 3$, one arrives at a contradiction similar to that in the proof of Theorem 4.8 .

<u>Theorem 4.11</u> : Let $L = F / F_{(n_1),\ldots,(n_k)}$, $n_i \geqslant 2$, $k \geqslant 2$, be a finitely-generated free polynilpotent Lie algebra and S be an ideal of L. If rank of S is less than the rank of L, then S cannot be free polynilpotent as a subalgebra relative to the same sequence as L.

<u>Proof</u> : The proof of this theorem is almost identical to that of Theorem 4.9 .

Note that if Y is a subset of $L = F / F_{(n_1),\ldots,(n_k)}$, which is linearly independent modulo $F_{(2)}$, then $\langle Y \rangle$ need not equal L even though $|Y| =$ rank of L. Hence we require rank of $S <$ rank of L in the statement of Theorem 4.11 . Furthermore since Center $(L) = \{0\}$, there are no non-trivial abelian ideals in L. (If S is an abelian ideal of L, then for any $f \in S$, there exists an element (take $x \in X$) such that $xf \notin F_{(n_1),\ldots,(n_k)}$ and hence $xf \neq 0$ in L, a contradiction)

Chapter 5

THE $(m + k)$-TH TERM OF THE LOWER CENTRAL SERIES OF A FREE LIE ALGEBRA AS A SUBALGEBRA OF THE m-TH TERM

In a free metabelian group G, $G_{(2)}$ and $G_{(3)}$ are free abelian groups and $G_{(3)}$ is a direct summand of $G_{(2)}$, since $G_{(2)} / G_{(3)}$ is also free abelian. For an absolutely free group F this is false, since $F_{(2),(2)} \subsetneq F_{(3)}$. However, one can find free generating sets for $F_{(2)}$ and $F_{(3)}$ which have a large number of elements in common.

M. Ward has looked at the following question in [25] : How close does $F_{(3)}$ come to being a free factor of $F_{(2)}$? The answer is the following :

"In an absolutely free group F of arbitrary rank there exists subgroups A_n and B_n, $n = 1,2,\ldots$ such that, for each non-negative integer n, $\delta^n F_{(2)}$ is the free product of A_n and B_n and $\delta^n F_{(3)}$ is the free product of B_n and A_{n+1}."

In this chapter we look at a more general form of the same problem for Lie algebras ; we consider $F_{(m+k)}$ as a subalgebra of $F_{(m)}$, where F is a free algebra.

§ 1. $F_{(m+k)}$ as a subalgebra of $F_{(m)}$, $k \leq m$.

The following result will be used throughout this chapter :

Lemma 5.1 : Let F be a free Lie algebra and M be a subalgebra of F. Suppose M has a well-ordered free generating set which is the union of two sets $Y \cup Z$ such that every element of Y precedes every element of Z. Let

$$Y' = \left\{ a_1(a_2(\dots(a_r b))\dots) \ : \ a_i \in Y, \ b \in Y \cup Z \ ; \ r \geq 1 \right.$$
$$\left. a_1 \geq a_2 \geq \dots \geq a_r < b \right\}$$
$$Z' = \left\{ a_1(a_2(\dots(a_r b))\dots) \ : \ a_i \in Z, \ b \in Z \cup Y' \ ; \ r \geq 1 \right.$$
$$\left. a_1 \geq a_2 \geq \dots \geq a_r < b \right\},$$

where we assume in the definition of Z' that the elements of $Y \cup Z$ precede those of Y'. Then the elements of $Y' \cup Z'$ are distinct as written and form a set of free generators for $\delta^1 M = M_{(2)}$.

<u>Proof</u> : M is a free Lie algebra with a free generating set $Y \cup Z$. By Theorem 1.9, the set D defined as

$$D = \left\{ a_1(a_2(\dots(a_r b))\dots) \ : \ a_i, b \in Y \cup Z \ ; \ r \geq 1 \right.$$
$$\left. a_1 \geq a_2 \geq \dots \geq a_r < b \right\}$$

is a set of free generators for $M_{(2)}$. It remains to show that $Y' \cup Z' = D$.

Let $g \in Y' \cup Z'$. If $g \in Y'$ and

$$g = a_1(a_2(\dots(a_r b))\dots)$$

as in the definition of Y', then obviously $g \in D$. Suppose $g \in Z'$ and

$$g = a_1(a_2(\dots(a_r b))\dots)$$

where $a_i \in Z$, $b \in Z \cup Y'$. If $b \in Z$, then obviously $g \in D$. Let us assume that $b \in Y'$. Then

$$b = c_1(c_2(\dots(c_s d))\dots)$$

where $c_i \in Y$, and $d \in Y \cup Z$. Since $a_r \in Z$ and $c_1 \in Y$, the order on the set $Y \cup Z$ implies that $a_r > c_1$. Hence the element

$$g = a_1(a_2(\dots(a_r(c_1(c_2(\dots(c_s d))\dots)$$

belongs to D. Therefore $Y' \cup Z' \subseteq D$.

Now, suppose $f \in D$, where

$$f = a_1(a_2(\ldots(a_r b))\ldots)$$

$a_i, b \in Y \cup Z$. If all $a_i \in Y$, then obviously $f \in Y'$. If all $a_i \in Z$, then $a_r < b$ implies that $b \in Z$ and $f \in Z'$.

Suppose some $a_i \in Y$ and others belong to Z. Since every element of Y precedes elements of Z, there exists an integer k, $1 \leqslant k < r$, such that

$$a_1, \ldots, a_k \in Z$$

$$a_{k+1}, \ldots, a_r \in Y$$

Then the element $g = a_{k+1}(a_{k+2}(\ldots(a_r b))\ldots)$ belongs to Y' and thus $f = a_1(a_2(\ldots(a_k g))\ldots)$ belongs to Z'. Therefore, $f \in Y' \cup Z'$ and $Y' \cup Z' \supseteq D$.

Obviously the elements of $Y' \cup Z'$ are distinct and thus they form a set of free generators for $M_{(2)} = \delta' M$.

Let F be a free Lie algebra with a free generating set X and H be a Hall basis for F. Suppose m and k are fixed positive integers, $2 \leqslant k \leqslant m$. An explicit form for the free generating set C_m of $F_{(m)}$ is given by

$$C_m = \left\{ \begin{array}{l} f = a_1(a_2(\ldots(a_r b))\ldots) \ : \ \text{length}\ (f) \geqslant m \ ; \ r \geqslant 1 \\ a_i,\ b \in (H_1 \cup \ldots \cup H_{m-1}) \ ; \ a_1 \geqslant \ldots \geqslant a_r < b \\ \text{and if } \ b = b_1 b_2 \ \text{ then } \ a_r \geqslant b_1 \end{array} \right\}$$

The inequality signs, of course, refer to the ordering in $(H_1 \cup \ldots \cup H_{m-1})$ defined in Chapter 1. Elements of C_m are ordered among themselves according to X-lengths and those with the same X-length are ordered arbitrarily. This gives C_m a length-preserving order.

We now partition C_m into two disjoint sets S_o and T_o where

$$S_o = H_m \cup \dots \cup H_{m+k-1}$$

$$T_o = \left\{ g \in C_m : \text{length}(g) \geqslant k+m \right\}$$

It was shown in Chapter 1, § 5 that $(H_m \cup \dots \cup H_{2m-1}) \subsetneq C_m$, Since $k \leqslant m$, obviously $S_o \subsetneq C_m$ and $C_m = S_o \cup T_o$. Also every element of S_o has smaller length than every element of T_o so that if $f \in S_o$ and $g \in T_o$ we put $f < g$ under the ordering described above. Now define

$$S_1 = \left\{ a_1(a_2(\dots(a_r b))\dots) : a_i \in S_o, \ b \in S_o \cup T_o ; \right.$$
$$\left. a_1 \geqslant \dots \geqslant a_r < b \ ; \ r \geqslant 1 \right\}$$

$$T_1 = \left\{ a_1(a_2(\dots(a_r b))\dots) : a_i \in T_o, \ b \in T_o \cup S_1 ; \right.$$
$$\left. a_1 \geqslant \dots \geqslant a_r < b \ ; \ r \geqslant 1 \right\}$$

In the definition of T_1, if $b \in T_o$ then we use the order in T_o to decide if $a_r < b$. If $b \in S_1$, then $b \in H_i^{C_m}$, $i \geqslant 2$. (H^{C_m} as described in Chapter 1). Obviously the sets $S_o \cup T_o$ and S_1 are disjoint, since $S_1 \subsetneq F_{(m),(2)}$ but $(S_o \cup T_o) \cap F_{(m),(2)} = \{0\}$. Hence we put every element of $S_o \cup T_o$ to be less than every element of S_1.

<u>Lemma 5.2</u> : $S_1 \cup T_1$ is a set of free generators for $\delta' F_{(m)}$.

<u>Proof</u> : It follows by using Lemma 2.1, putting $M = F_{(m)}$.

By Lemma 2.1, the sets S_1 and T_1 are disjoint. We give them on order such that every element of S_1 precedes every element of T_1.

We then define S_2 and T_2 so that $S_2 \cup T_2$ will be a set of free generators for $\delta^1(\delta^1 F_{(m)}) = \delta^2 F_{(m)}$.

Suppose the sets $S_1, T_1, \dots, S_{n-1}, T_{n-1}$ are defined and well-ordered in such a way that every element of S_i precedes every element of T_i for $i = 1, \dots, n-1$, and elements of $S_{i-1} \cup T_{i-1}$ precede the elements of $S_i \cup T_i$, which can be done since the elements of $S_i \cup T_i$ are products of two or more elements from $S_{i-1} \cup T_{i-1}$.

In particular, suppose $S_{n-1} \cup T_{n-1}$ is defined and well-ordered and forms a set of free generators for $\delta^{(n-1)} F_{(m)}$. Then, we define

$$S_n = \left\{ a_1(a_2(\dots(a_r b))\dots) \;:\; a_i \in S_{n-1},\; b \in S_{n-1} \cup T_{n-1} ;\; a_1 \geq \dots \geq a_r < b \;;\; r \geq 1 \right\}$$

$$T_n = \left\{ a_1(a_2(\dots(a_r b))\dots) \;:\; a_i \in T_{n-1},\; b \in T_{n-1} \cup S_n ;\; a_1 \geq \dots \geq a_r < b \;;\; r \geq 1 \right\}$$

By Lemma 5.1, $S_n \cup T_n$ is a set of free generators for $\delta^n F_{(m)}$. Furthermore, elements of $S_n \cup T_n$ are distinct as written. We well-order this set in such a way that every element of S_n precedes every element of T_n.

<u>Definition 5.1</u> : Let $g \in F$ such that

$$g = a_1(a_2(\dots(a_r b))\dots)$$

, where a_i , $b \in H$, $a_1 \geq a_2 \geq \dots \geq a_r < b$, $r \geq 1$ and if $b = b_1 b_2$, then $a_r \geq b_1$. We call g an element of the standard form. The inequalities in the definition refer to the order in H.

In particular, C_m, the free generating set for $F_{(m)}$, consists of elements in H of the standard form which have length $\geq m$.

<u>Lemma 5.3</u> : $T_o \cup S_1$ is a set of free generators for $F_{(m+k)}$, $k \leqslant m$.

<u>Proof</u> : Obviously $T_o \cap S_1 = \{0\}$, since the elements of S_1 have C_m-length $\geqslant 2$, whereas the elements of T_o have C_m-length $= 1$. It remains to show that $T_o \cup S_1 = C_{m+k}$.

Let $g \in T_o \cup S_1$. If $g \in T_o$, then length $(g) \geqslant m+k$ and $g = a_1(a_2(\ldots(a_r b))\ldots)$ is of the standard form, where a_i and b belong to $(H_1 \cup \ldots \cup H_{m-1})$. Thus $g \in C_{m+k}$. Now suppose $g \in S_1$,

$$g = a_1(a_2(\ldots(a_r b))\ldots) \quad ,$$

where $a_i \in S_o$ and $b \in S_o \cup T_o$. If $b \in S_o$, then from the definition of S_o we have

$$\text{length } (g) \geqslant 2m \geqslant m + k$$

and g is of the standard form. Thus g is a member of C_{m+k}. On the other hand if $b \in T_o$, then

$$b = c_1(c_2(\ldots(c_s d))\ldots)$$

is of the standard form, where $c_i, d \in (H_1 \cup \ldots \cup H_{m-1})$ and length $(b) \geqslant m + k$. Then

$$g = a_1(a_2(\ldots(a_r(c_1(c_2(\ldots(c_s d))\ldots)$$

is of the standard form, since $a_r \in S_o$ implies that

$$\text{length } (a_r) \geqslant m > \text{length } (c_1)$$

Thus $g \in C_{m+k}$ and $T_o \cup S_1 \subseteq C_{m+k}$.

Conversely, let $z \in C_{m+k}$

$$z = a_1(a_2(\ldots(a_r b))\ldots)$$

be of the standard form, where $a_i, b \in (H_1 \cup \ldots \cup H_{m+k-1})$. If for $i = 1, \ldots r$, each a_i belongs to $(H_1 \cup \ldots \cup H_{m-1})$, then $z \in T_o$. On the other hand, if each a_i and b belong to $(H_m \cup \ldots \cup H_{m+k-1})$, then $a_i, b \in S_o$ and $z \in S_1$.

Now suppose that some a_i belong to $(H_m \cup \ldots \cup H_{m+k-1})$ and others belong to $(H_1 \cup \ldots \cup H_{m-1})$. Since $a_1 \geqslant \ldots \geqslant a_r$, there exists an integer t, $1 \leqslant t < r$, such that

$$a_1, \ldots, a_t \in (H_m \cup \ldots \cup H_{m+k-1})$$

$$a_{t+1}, \ldots, a_r \in (H_1 \cup \ldots \cup H_{m-1})$$

Consider the word y defined by

$$y = a_{t+1}(a_{t+2}(\ldots(a_r b))\ldots)$$

which is of the standard form. Then

$$z = a_1(a_2(\ldots(a_t y))\ldots) .$$

Since $z \in C_{m+k}$, $a_t < y$ and thus

$$\text{length } (y) \geqslant \text{length } (a_t) \geqslant m$$

If $m \leqslant \text{length } (y) \leqslant m + k + 1$, then $y \in S_o$, each $a_i \in S_o$ for $i = 1, \ldots, t$ and thus $z \in S_1$. If length $(y) \geqslant m + k$, then $y \in T_o$ which implies that $z \in S_1$. Hence $C_{m+k} \subseteq T_o \cup S_1$.

Therefore, $C_{m+k} = T_o \cup S_1$.

The elements of T_o precede the elements of S_1. By definition of T_1 and S_2, we have by Lemma 5.1 that $T_1 \cup S_2$ form a set of free generators for $\delta^1 F_{(m+k)}$. Suppose that $T_{n-1} \cup S_n$ is a set of free generators for $\delta^{n-1} F_{(m+k)}$. Then, by induction and Lemma 5.1, $T_n \cup S_{n+1}$ is a set of free generators for $\delta^n F_{(m+k)}$.

<u>Definition 5.2</u> : Let $\{F^n\}$ be a family of free Lie algebras

indexed by the positive integers $n \in N$, each one defined over the same field $\underline{k}$. Then a Lie algebra F over $\underline{k}$ is said to be the free Lie sum of the $\{F^n\}$ if the $\{F^n\}$ form a family of subalgebras of F such that

(i) $F^n \cap F^m = \{0\}$, $m \neq n$

(ii) if X^n is a set of free generators for F^n, then $\bigcup_n X^n$ is a set of free generators for F.

Let A_n, B_n be subalgebras of F freely generated by S_n and T_n respectively. $S_n \cup T_n$ is a set of free generators for $\delta^n F_{(m)}$. Hence $\delta^n F_{(m)}$ is the free Lie sum of A_n and B_n. Similarly $\delta^n F_{(m+k)}$ is the free Lie sum of B_n and A_{n+1}.

Theorem 5.1 : Let F be a free Lie algebra over a field $\underline{k}$. Then there exists subalgebras A_n and B_n, for $n = 0,1,\ldots$, such that for each n, $\delta^n F_{(m)}$ is the free Lie sum of A_n and B_n and $\delta^n F_{(m+k)}$ is the free Lie sum of B_n and A_{n+1}, where $k \leq m$.

---

§ 2. $F_{(m+k)}$ As a Subalgebra of $F_{(m)}$, $k > m$

Theorem 5.1 fails to be true for $k > m$. Suppose we define the sets S_n, T_n for $n = 0,1,\ldots$, as in the previous section. Then $S_n \cup T_n$ is a set of free generators for $\delta^n F_{(m)}$, but $T_0 \cup S_1$ fails to form a set of free generators for $F_{(m+k)}$, $k > m$. Let $g \in S_1$ such that

$$g = a_1 b$$

where a_1, $b \in S_0$ and length (a_1) = length $(b) = m$. Then

$$\text{length } (g) = 2m < m + k$$

and $g \notin C_{m+k}$. Furthermore, there are elements in S_2 which belong to C_{m+k}.

For some positive integer t, one can define subsets S_i' of S_i for $i = 1,\ldots,t$ such that

$$(*) \qquad \left(\bigcup_{i=1}^{t} S_i' \right) \cup T_o$$

generates $F_{(m+k)}$ freely. However, the sets constructed on (*) do not form a free generating set for $\delta^1 F_{(m+k)}$, as described above.

APPENDIX

Comparison of the Notation Used in This Thesis with That Used in [23] and [24]

Let F be a free Lie algebra on the free generating set X and M be a free group on the same free generating set X. We put

$$L = F / F_{(n_1),\dots,(n_k)}$$

$$G = G / G_{(n_1),\dots,(n_k)}$$

to denote the free polynilpotent Lie algebra and the free polynilpotent group relative to the sequence $\{ n_1,\dots,n_k \}$ respectively. Then we can compare the notation used in this thesis with that used in [23] and [24] as follows :

Notation of this thesis		M. Ward's notation as in [23] and [24]
Basis H of F	—	Basis $\underline{H}$ of M
Basis B of L	—	Basis $\underline{B}$ of G
$X \cup H_2 \cup \dots \cup H_{n_1-1}$	—	Elements in $\underline{B}$ of depth 0
H_i	—	Elements in $\underline{B}$ of depth 0 and 0-weight i for $1 \leqslant i < n_1$
$H_1^{Cn_1,\dots,n_j} \cup \dots \cup H_{n_{j+1}-1}^{Cn_1,\dots,n_j}$	—	Elements in $\underline{B}$ of depth j, $j < k$
$H_i^{Cn_1,\dots,n_j}$	—	Elements in $\underline{B}$ of depth j and j-weight i, $i < n_{j+1}$

Order on B is determined by the "generalized length" length of an element	π, the fine order associated with the polyweight range Q.
$C_{n_1,\dots,n_j}$-length (x), for $x \in B$	$\pi(x)(j)$, the j-weight of x for $x \in \underline{B}$
X-length (x), $x \in B$	$\pi(x)(0)$, $x \in \underline{B}$
Order on B is determined by the following in decreasing order of priority : $C_{n_1,\dots,n_{k-1}}$-lengths $C_{n_1,\dots,n_{k-2}}$-lengths $\vdots$ C_{n_1}-lengths X-lengths	Order on $\underline{B}$ is determined by the following in decreasing order of priority : $\pi(x)(k-1)$ $\pi(x)(k-2)$ $\vdots$ $\pi(x)(1)$ $\pi(x)(0)$ for $x \in \underline{B}$

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